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Objekttyp: Article

Zeitschrift: L'Enseignement Mathématique

# Band (Jahr): 18 (1972)

# Heft 1: L'ENSEIGNEMENT MATHÉMATIQUE

PDF erstellt am: 25.05.2024

Persistenter Link: https://doi.org/10.5169/seals-45370

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# ON IDEAL-ADIC TOPOLOGIES FOR A COMMUTATIVE RING

by Robert GILMER

Let R be a commutative ring and let A and B be ideals of R such that  $B \subseteq A$ . We wish to consider the relationship between the following two conditions:

1) R is complete in the A-adic topology <sup>1</sup>).

2) R is complete in the B-adic topology.

Several results in this direction are known; for example:

THEOREM 1. ([4, Theorem 14, p. 275]) Assume that R is Noetherian with identity, and that R is complete Hausdorff in its A-adic topology. Then R is complete in its B-adic topology.

In [3], M. O'Malley proves the following theorem.

THEOREM 2. If R has an identity, and if R is complete Hausdorff in its A-adic topology, then R is complete in the (b)-adic topology for each element b of A.

In [2, Theorem 2.1 and Corollary 2.2], O'Malley extends his results in [3] to prove:

THEOREM 3. If R contains an identity, if  $A = (a_1, ..., a_n)$  is finitely generated, and if R is Hausdorff in the A-adic topology, then R is complete in its A-adic topology if and only if R is complete in its  $(a_i)$ -adic topology for each i between 1 and n.

COROLLARY 1. If R contains an identity, and if R is a complete Hausdorff space in its A-adic topology, then R is complete Hausdorff in its B-adic topology for each finitely generated ideal B contained in A.

Moreover, O'Malley observes in [2] that Theorem 2, Theorem 3, and Corollary 1 are true without the assumption that R contains an identity, for the following result is valid.

<sup>&</sup>lt;sup>1</sup>) i.e. the topology for which a fundamental system of neighbourhoods of 0 is  $A, A^2, A^3, ...$ 

PROPOSITION 1. Assume that R is a commutative ring, and S is a ring obtained by the canonical adjunction of an identity of characteristic zero to R (see [1, p. 4]). Let A be an ideal of R. Then A is an ideal of S and

1) R is Hausdorff in its A-adic topology if and only if S is Hausdorff in its A-adic topology;

2) R is complete in its A-adic topology if and only if S is complete in its A-adic topology.

O'Malley obtains the results we have cited from a much deeper theory of the set of *R*-endomorphisms of the power series ring R[[X]]. Our purpose here is to obtain O'Malley's results from basic topological considerations, independent of the theory of *R*-endomorphisms of R[[X]].

PROPOSITION 2. Assume that  $\{A_i\}_{i=1}^n$  is a finite set of ideals of the commutative ring R, and let  $A = A_1 + ... + A_n$ . If R is complete in its  $A_i$ -adic topology for each i between 1 and n, then R is complete in its A-adic topology.

*Proof.* We note that the A-adic topology on R is the topology induced by the sequence  $\{B_i\}_{i=1}^{\infty}$  of ideals, where  $B_i = A_1^i + \ldots + A_n^i$ . This is true because  $A^i \supseteq B_i \supseteq A^{ni}$  for each positive integer *i*. Thus, if  $\{c_i\}_0^{\infty}$  is a Cauchy sequence in the A-adic topology, then by passage to a subsequence of  $\{c_i\}_0^{\infty}$ , we can assume that  $c_i - c_{i-1} \in B_i$  for each positive integer *i*. If we write  $c_i - c_{i-1} = a_{1i} + a_{2i} + \ldots + a_{ni}$ , where  $a_{ji} \in A_j^i$ , then for each *i*,

$$c_i = c_0 + \sum_{j=1}^n \sum_{k=1}^i a_{jk}$$
.

The series  $\sum_{k=1}^{\infty} a_{jk}$  converges in the  $A_j$ -adic topology; we let  $a_j^* = \lim_{k} (a_{j1} + a_{j2} + ... + a_{jk})$  Then it is clear that the sequence  $\{c_i\}_{0}^{\infty}$  converges to  $c_0 + \sum_{j=1}^{n} a_j^*$  in the A-adic topology. Therefore R is complete in its A-adic topology.

We remark that in Proposition 2, the A-adic topology on R need not be Hausdorff, although the  $A_i$ -adic topology is Hausdorff for each *i*. For example, if k is a field, then k [[X, Y, Z]]/A, where A = (Z(1-X-Y)), is complete Hausdorff under its [(X) + A]/A-adic and [(Y) + A]/A-adic topologies, but is not Hausdorff under its [(X, Y) + A]/A-adic topology.

THEOREM 4. Assume that R is a commutative ring, and that R is a complete Hausdorff space in its A-adic topology. If  $b \in A$ , then R is complete in its (b)-adic topology.

*Proof.* The (b)-adic topology on R is equivalent to the topology induced on R by the sequence  $\{B_i\}_{i=1}^{\infty}$  of ideals, where  $B_i = Rb^i$ . This is true since  $(b^i) \supseteq B_i \supseteq (b^{i+1})$  for each *i*. To prove that R is complete in its (b)-adic topology, it suffices to show that each sequence  $\{c_i\}_{i=0}^{\infty}$ , where  $c_i - c_{i-1} \in B_i$  for each *i*, converges in the (b)-adic topology. Since  $b \in A$ , the sequence  $\{c_i\}$  converges to an element  $c^*$  in the A-adic topology. We prove that  $c_i$  converges to  $c^*$  in the (b)-adic topology. Thus if  $c_i - c_{i-1} = r_i b^i$ for each positive integer *i*, then for positive integers *k* and *n*, we have

$$c_{k+n} - c_k = b^{k+1} [r_{k+1} + r_{k+2}b + \dots + r_{k+n}b^{n-1}].$$

Taking limits in the A-adic topology as n approaches infinity, and using the fact that the A-adic topology is Hausdorff, we obtain

$$c^* - c_k = b^{k+1} s^*_{k+1}$$
 where  $s^*_{k+1} = \sum_{n=1}^{\infty} r_{k+n} b^{n-1}$ .

It follows that  $c^* - c_k \in B_t$  for each  $k \ge t - 1$ , and  $\{c_i\}$  converges to  $c^*$  in the (b)-adic topology, as asserted.

Theorem 4 fails if the assumption that R is Hausdorff in the A-adic topology is dropped. For example, if R is idempotent, then R is complete in its R-adic topology, but R need not be complete in its (b)-adic topology for each b in R. For a less trivial example,  $Z \oplus Z$  is complete in its  $(Z \oplus (0))$ -adic topology, but not in its  $((2) \oplus (0))$ -adic topology.

Proposition 2 and Theorem 4 yield alternate proofs of Theorems 2 and 3 and Corollary 1 (dropping, in each case, the assumption that R has an identity).

We remark that in general, R need not be complete in its B-adic topology if R is complete Hausdorff in its A-adic topology, even if A is principal. Thus let D be an integral domain with identity containing a prime ideal  $C = (c_1, c_2, ..., c_n, ...)$  such that C is countably generated, but C is not the radical of a finitely generated ideal. (For example, let  $D = J[\{X_i\}_{i=1}^{\infty}]$ , where J is an integral domain with identity and let  $C = (\{X_i\}_{i=1}^{\infty}]$ .) The ring R = D[[Y]] is a complete Hausdorff space in its (Y)-adic topology. But if  $B = (\{cY \mid c \in C\})$ , then R is not complete in the B-adic topology, for  $\{c_1Y, c_1Y + c_2^2Y^2, ...\}$  is a Cauchy sequence in the B-adic topology which converges to  $f = \sum_{i=1}^{\infty} c_i^i Y^i$  in the (Y)-adic topology. If this sequence converges in the B-adic topology, it must converge to f. But

$$f - \left(\sum_{1}^{n} c_{i}^{i} Y^{i}\right) = \sum_{n+1}^{\infty} c_{i}^{i} Y^{i} \notin B$$

for each positive integer, for if  $\sum_{n+1}^{\infty} c_i^i Y^i \in B$ , then for some positive integer k,  $\sum_{n+1}^{\infty} c_i^i Y^i \in (c_1 Y, ..., c_k Y)$ , and  $c_i^i \in (c_1, ..., c_k)$  for each  $i \ge n + 1$ .

It follows that  $C = \sqrt{(c_1, ..., c_k)}$ , contrary to our assumptions concerning C.

Added in proof. Matthew O'Malley has pointed out to the author that in the remark preceding Theorem 4, the ring k[[X, Y, Z]]/A is Hausdorff in its [(X, Y) + A]/A-adic topology.

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(Reçu le 20 janvier 1972)

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