LEBESGUE GENERATING MEASURES

by Gerald Freilich

1. *Introduction.* The treatment of the theory of Lebesgue measure in Euclidean n-dimensional space $\mathbf{R}^n$ frequently begins with the assignment of " elementary measure " to rectangular parallelepipeds with equal edges parallel to the coordinate axes (hereafter called n-cubes), the measure assigned being the n^{th} power of the common edge length. Then by considering the infimum of the sums of elementary measures of countable coverings by n-cubes, an outer measure is assigned to each subset of $\mathbf{R}^n$ and the theory of outer measure, measurable sets and measure is developed.

A natural question to ask in the above development is the following: Suppose the process begins with the assignment of " elementary measure " to some collection of sets other than the n-cubes, will the resulting measure equal Lebesgue measure? Examples of such collections include the set of all n-spheres, and the set of all n-cubes rotated through a fixed angle. The question will be investigated in this paper and results will be obtained which guarantee under general conditions that the process generates Lebesgue measure. In particular, our results imply certain well-known theorems of Hausdorff and of Sard.

2. *Generating Measures.* For the remainder of this paper,

L will denote Lebesgue outer measure in $\mathbf{R}^n$,

$\mathscr{F}$ will denote a family of subsets of R^n with $\emptyset \in \mathscr{F}$,

M will denote a non-negative function defined on $\mathscr{F}$ with $M(\emptyset) = 0$.

Corresponding to $\mathscr{F}$ and M, two processes for generating outer measures $\varphi_{\mathscr{F},M}$ and $\Phi_{\mathscr{F},M}$ will be defined; the process involved in generating $\varphi_{\mathscr{F},M}$ is the formalization of the process described in the Introduction.

Definition. Given $\mathscr{F}$ and M, the outer measures $\varphi_{\mathscr{F},M}$ and $\Phi_{\mathscr{F},M}$ are defined by the formulas:

$$\varphi_{\mathscr{F},M}(A) = \inf\Big(\sum_{i=1}^{\infty} M(\alpha_i) \mid A \subset \bigcup_{i=1}^{\infty} \alpha_i,\ \alpha_i \in \mathscr{F} \Big) \quad \text{for} \quad A \subset R^n,$$

$$\Phi_{\mathscr{F},M}(A) = \lim_{\delta \to 0+} \inf\Big(\sum_{i=1}^{\infty} M(\alpha_i) \mid A \subset \bigcup_{i=1}^{\infty} \alpha_i,\ \alpha_i \in \mathscr{F},\ \operatorname{diam} \alpha_i < \delta \Big) \quad \text{for} \quad A \subset R^n.$$

(Note: If no countable collection of sets of $\mathscr{F}$ covers A, then

$$\varphi_{\mathscr{F},M}(A) = \Phi_{\mathscr{F},M}(A) = \infty.)$$

THEOREM 1. $\Phi_{\mathscr{F},M}$ is a countably additive measure when restricted to the Borel subsets of R^n.

Remark. Theorem 1 is well-known [2, p. 105). Note however that $\varphi_{\mathscr{F},M}$ need not be a measure when restricted to the Borel sets in R^n.

THEOREM 2. $\varphi_{\mathscr{F},M} \leqslant \Phi_{\mathscr{F},M}$ and if $M \geqslant L$ on $\mathscr{F}$, then $L \leqslant \varphi_{\mathscr{F},M} \leqslant \Phi_{\mathscr{F},M}$.

Proof. Since for $\delta > 0$,

$$\varphi_{\mathscr{F},M}(A) \leqslant \inf\Big(\sum_{i=1}^{\infty} M(\alpha_i) \mid A \subset \bigcup_{i=1}^{\infty} \alpha_i,\ \alpha_i \in \mathscr{F},\ \operatorname{diam} \alpha_i < \delta\Big),$$

it follows that $\varphi_{\mathscr{F},M}(A) \leqslant \Phi_{\mathscr{F},M}(A)$.

Next assume $M(\alpha) \geqslant L(\alpha)$ for all $\alpha \in \mathscr{F}$. If each $\alpha_i \in \mathscr{F}$ and $A \subset \bigcup_{i=1}^{\infty} \alpha_i$, it follows that

$$L(A) \leqslant \sum_{i=1}^{\infty} L(\alpha_i) \leqslant \sum_{i=1}^{\infty} M(\alpha_i),$$

and hence that

$$L(A) \leqslant \varphi_{\mathscr{F},M}(A).$$

THEOREM 3. If $\varnothing \in \mathscr{G} \subset \mathscr{F}$, then $\varphi_{\mathscr{F},M} \leqslant \varphi_{\mathscr{F},M}$ and $\Phi_{\mathscr{F},M} \leqslant \Phi_{\mathscr{F},M}$.

Proof. Obvious.

THEOREM 4. If $\mathscr{F}$ is the set of all n-cubes of R^n and $M = L$, then

$$L = \varphi_{\mathscr{F},M} = \Phi_{\mathscr{F},M}.$$

Proof. By definition, $L(A) = \varphi_{\mathscr{F},M}(A)$ for $A \subset R^n$. Since any n-cube can be subdivided into subcubes of arbitrarily small diameter but of the same total n-volume, it follows that $\varphi_{\mathscr{F},M}(A) = \Phi_{\mathscr{F},M}(A)$.

Remark. Theorem 4 remains true if $\mathscr{F}$ is the set of all open n-cubes of R^n, or if $\mathscr{F}$ is the set of all closed n-cubes.

3. *Lebesgue Generating Measures.* The main result is the following:

THEOREM 5. Let α be a bounded subset of R^n, Interior $\alpha \neq \varnothing$, $L(\bar{\alpha} - \alpha) = 0$, where $\bar{\alpha}$ is the closure of α. Define $\mathscr{G}$ as the set containing $\varnothing$ and all images of α under translations and homothetic transformations

$((x_1, \ldots, x_n) \mapsto (kx_1, \ldots, kx_n), k > 0)$ of R^n. If $\mathscr{G} \subset \mathscr{F}$, $M \geqslant L$ on $\mathscr{F}$ and $M = L$ on $\mathscr{G}$, then

$$\varphi_{\mathscr{F},M} = \Phi_{\mathscr{F},M} = L.$$

Proof. By Theorem 2, $L \leqslant \varphi_{\mathscr{G},M} \leqslant \Phi_{\mathscr{G},M}$. We shall prove that $\Phi_{\mathscr{G},M} = L$. To accomplish this, choose an open n-cube $\Delta \subset \alpha$, $\Delta \neq \emptyset$. Let $k = L(\alpha)(L(\Delta))^{-1}$, $j = \operatorname{diam}(\alpha)(\operatorname{diam}(\Delta))^{-1}$. If $A \subset R^n$ and $d > 0$, then any covering of A by open n-cubes $\{\Delta_i\}$ of diameter less than d can be replaced by a covering of A by sets $\{\alpha_i\}$ in $\mathscr{G}$ with diameter less than jd and such that $\sum_1^\infty M(\alpha_i) = \sum_1^\infty L(\alpha_i) = k \sum_1^\infty L(\Delta_i)$. It follows that $\Phi_{\mathscr{G},M}(A) \leqslant k\,L(A)$ by Theorem 4. Since $\Phi_{\mathscr{G},M}$ is translation invariant, it follows from [3, p. 50] that for some $1 \leqslant p < \infty$, $\Phi_{\mathscr{G},M} = p \,.\, L$ on the Borel sets of R^n. In particular, $L(A) = 0$ if and only if $\Phi_{\mathscr{G},M}(A) = 0$.

Next let c be a closed unit n-cube in R^n. Enclose c in a closed n-cube C with the same center as c but with sides of length $1 + 2\delta$, where $\delta > 0$ is arbitrary. Apply the Vitali theorem [4, p. 109] to the set of all $\bar{\beta}$ where $\beta \in \mathscr{G}$, $\operatorname{diam}\beta < \delta$, $\beta \cap c \neq \emptyset$ to obtain adisjoint countable collection $\{\bar{\beta}_i \mid i = 1, 2, \ldots\}$ such that

$$L(c - \bigcup_{i=1}^{\infty} \bar{\beta}_i) = 0.$$

Since $L(\bar{\beta}_i - \beta_i) = 0$ for each i, it follows that $L(c - \bigcup_{i=1}^{\infty} \beta_i) = 0$ and hence that $\Phi_{\mathscr{G},M}(c - \bigcup_{i=1}^{\infty} \beta_i) = 0$. Since $\Phi_{\mathscr{G},M}(c - \bigcup_{i=1}^{\infty} \beta_i) = 0$, it follows that $c - \bigcup_{i=1}^{\infty} \beta_i$ can be covered by a countable number of sets $\gamma_i \in \mathscr{G}$, $\operatorname{diam}\gamma_i < \delta$ and such that $\sum_{i=1}^{\infty} L(\gamma_i) = \sum_{i=1}^{\infty} M(\gamma_i) < \delta$. Since $\bigcup_{i=1}^{\infty} \beta_i \subset C$ and the sets β_i are disjoint, it follows that

$$\sum_{i=1}^{\infty} M(\beta_i) = \sum_{i=1}^{\infty} L(\beta_i) \leqslant L(C) = (1+2\delta)^n.$$

Now $c \subset \bigcup_{i=1}^{\infty} \beta_i \cup \bigcup_{i=1}^{\infty} \gamma_i$, $\operatorname{diam}\beta_i < \delta$, $\operatorname{diam}\gamma_i < \delta$ for all i, and

$$\sum_{i=1}^{\infty} M(\beta_i) + \sum_{i=1}^{\infty} M(\gamma_i) \leqslant (1+2\delta)^n + \delta.$$

Allowing δ to approach zero, we conclude that $\Phi_{\mathscr{G},M}(c) \leqslant 1 = L(c)$. Hence $\Phi_{\mathscr{G},M}(c) = 1 = L(c)$ and therefore $p = 1$. This implies $\Phi_{\mathscr{G},M} = L$

on the Borel sets of R^n. Since for any set $A \subset R^n$, there exists a G_δ set $B \supset A$ with $L(B) = L(A)$, it follows that

$$L(A) \leqslant \Phi_{\mathscr{G},M}(A) \leqslant \Phi_{\mathscr{G},M}(B) = L(B) = L(A),$$
$$\Phi_{\mathscr{G},M}(A) = L(A).$$

Having proved $\Phi_{\mathscr{G},M} = L$, we apply Theorems 2 and 3 to conclude

$$L \leqq \varphi_{\mathscr{F},M} \leqq \Phi_{\mathscr{F},M} \leqq \Phi_{\mathscr{G},M} = L,$$

and the proof is complete.

Corollary 1. [1, p. 163]. Hausdorff n-dimensional sphere outer measure in R^n agrees with L on all the subsets of R^n.

Proof. Recall that Hausdorff n-dimensional sphere outer measure is the outer measure $\Phi_{\mathscr{F},L}$ where $\mathscr{F}$ is the set of all spheres in R^n. Application of Theorem 5 is immediate.

Corollary 2. [5]. Hausdorff n-measure in R^n agrees with L on all subsets of R^n.

Proof. Recall that Hausdorff n-measure is the outer measure $\Phi_{\mathscr{F},M}$ where $\mathscr{F}$ is the set of all subsets of R^n and $M(A) = (n!)^{-1}\,\Gamma(\tfrac{1}{2})^{n-1}\,\Gamma\left(\dfrac{n+1}{2}\right)(\operatorname{diam} A)^n$. In Theorem 5, take α to be an n-sphere. The isodiametric inequality implies that $M \geqslant L$ on $\mathscr{F}$ and $M = L$ on $\mathscr{G}$. Application of Theorem 5 completes the proof.

Corollary 3. L is invariant under rotation.

Proof. For a fixed rotation, merely take $\mathscr{F}$ to be the set of rotated n-cubes and $M = L$.

The author wishes to acknowledge the interesting correspondence with Professor Arthur B. Brown which initiated this paper.

REFERENCES

[1] Hausdorff, F. Dimension und ausseres Mass, *Math. Ann.*, 79 (1918), pp. 157-179.

[2] Munroe, M. E. *Introduction to Measure and Integration*, Addison-Wesley, Reading, Mass., 1953.

[3] Rudin, W. *Real and Complex Analysis*, McGraw-Hill, N.Y., 1966.

[4] Saks, S. *Theory of the Integral*, Warsaw, 1937.

[5] Sard, A. The Equivalence of n-measure and Lebesgue Measure in E_n, *Bull. Amer. Math. Soc.*, 49 (1943), pp. 758-759.

(Reçu le 21 mars 1972)

Gerald Freilich
Queens College
City University of New York
Flushing, New York 11367