3. The Main Theorem

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3. The Main Theorem

All that has been said here so far was known long before the advent of Non-Standard Analysis. Now we come to the heart of the matter—the key theorem. It was first obtained by Robinson as a corollary to the so-called Compactness Theorem of mathematical logic. Later proofs were given by means of the ultraproduct construction which also has its roots in mathematical logic. We shall content ourselves with a mere statement of the result. R as usual denotes the real number system and N the natural number system.

Theorem 3.1 (Main theorem). There is a set R^* for which all of the following hold:

1. R is a proper subset of R^*.
2. To each n-place function $f(x_1, ..., x_n)$ from R^n to R ($n \geqq 1$), there corresponds a certain function $f^*(x_1, ..., x_n)$ from $(R^*)^n$ to R^* which agrees with $f(x_1, ..., x_n)$ on R^n.
3. To each n-place relation $A(x_1, ..., x_n)$ on R ($n \geqq 1$), there corresponds a certain relation $A^*(x_1, ..., x_n)$ on R^* which agrees with $A(x_1, ..., x_n)$ on R. The relation corresponding to the equality relation on R is the equality relation on R^*.
4. Every statement $\mathcal{S}$ formulated in terms of

   i) particular (fixed) real numbers
   ii) particular (fixed) real functions
   iii) particular (fixed) real relations
   iv) variables ranging over R
   v) logical operations and quantifiers

   is true about R if and only if the statement $\mathcal{S}^*$ obtained from it by

   a) replacing each $f(x_1, ..., x_n)$ by $f^*(x_1, ..., x_n)$
   b) replacing each $A(x_1, ..., x_n)$ by $A^*(x_1, ..., x_n)$
   c) letting the variables range over R^*

   is true about R^*.

It turns out that there are many such R^*. From here on out it will be assumed that we are fixing on one of them.

The theorem is quite a mouthful and it must be admitted that our formulation of it suffers from a little imprecision owing to the fact that we never said what a statement is [1]). A few examples, however, should nail the idea down. Let us add for emphasis that we are only allowing statements of finite length.

Example 3.1. Consider the statement

$$(\forall x)\,(0+x=x)$$

which is true when the variables range over R; it asserts that the particular real number 0 is a left identity for the $+$ operation. By the Main Theorem, the statement

$$(\forall x)\,(0+^{*}\ x=x)$$

must be true when the variables range over R^*; thus 0 is also a left identity for the $+^*$ operation on R^*.

Example 3.2. Let f be a particular function from R to R which is an "onto" function. Then the statement

$$(\forall y)\,(\exists x)\,(f(x)\;=\;y)$$

is true when the variables range over R. Therefore by the Main Theorem the statement

$$(\forall y)\,(\exists x)\,(f^{*}(x)\;=\;y)$$

is true when the variables range over R^*; that is, the function f^* is onto R^*.

Henceforth instead of saying "true when the variables range over R", we shall simply say "true in R".

In subsequent discussions members of R will be called *standard* numbers, while members of $R^* - R$ will be called *non-standard* numbers. Likewise functions from R^n to R ($n \geqq 1$), relations on R, and subsets of R will be called *standard* functions, relations and subsets. Some writers refer to members of R^* as real numbers, but we shall reserve the term for members of R. Thus standard number and real number have the same meaning here.

Statements which can be formulated in the manner prescribed in the hypothesis of the Main Theorem are called *admissible* statements. You should convince yourself, by writing them out if necessary, that all the axioms of an ordered field are admissible; moreover, they are true about R (because

[1]) Using the terminology of formal logic the class of statements in question can be defined as the class of closed well-formed formulae of a generalized first-order language having distinct individual, function, and relation constants corresponding to each real, real function and real relation.

R is an ordered field). Now by the Main Theorem they are all true about R^* if we put the stars on the symbols $+$, $\times$, $<$. But this is just a way of saying that R^* is an ordered field with respect to $+^*$, $\times^*$, $<^*$. Moreover since the theorem provided that these agree with $+$, $\times$, $<$ respectively on R, we can say that R^* is an ordered field which has R as a proper subordered field. Now recalling results from our review on ordered fields we have that R^* is non-Archimedean and is not complete.

Now at this point you might be getting a bit suspicious. You might ask: "Why not show the completeness of R^* (and thus get a paradox) by taking the assertion that R is complete, and then use the Main Theorem to conclude that R^* is complete?" The catch is that the Completeness Axiom has a logical structure fundamentally different from the ordered field axioms. It's not an admissible statement! Its form is

$$(\forall S)\,(S \text{ bounded} \rightarrow \cdots\cdots\cdots)$$

that is, it has a variable ranging over the family of *subsets* of R. Recall, the variables in an admissible statement must range over R.

With respect to the Archimedean property the catch is a little different. Using the symbols $N(y)$ to denote the particular one-place relation—"y is a natural number," we *can* assert that R is Archimedean by the admissible statement

$$(\forall x)\,(\exists y)\,(N(y) \wedge x < y);$$

thus

$$(\forall x)\,(\exists y)\,(N^*(y) \wedge x <^* y)$$

is true in R^*, but it doesn't necessarily say that R^* is Archimedean. The y which is asserted to exist, and for which $N^*(y)$ holds, might be in $R^* - R$; that is, it might be non-standard. To be sure, it does say that R^* has some sort of formal Archimedean-like property, but if in the definition of Archimedean one requires that y actually be a member of N (and we shall), then R^* isn't Archimedean.

In the sequel it may at times be too repetitious to write statements first without the stars $*$, and then with them. It will usually be clear from the context whether the stars are intended. Thus if we were to say that

$$(\forall x)\,(\forall y)\,(x < y \rightarrow f(x) < f(y))$$

is true in R^*, then you are to understand that we are really talking about

$<^*, f^*$ and the variables are to range over R^*. Sometimes we shall put on some of the stars for emphasis.

4. Fixed Subsets

Let S be a particular (fixed) subset of R. We can identify S with the one-place relation $S(x)$ which holds for a given x if and only if $x \in S$; that is,

$$S = \{ x \in R \mid S(x) \}.$$

We can now define a set $S^* \subseteq R^*$ by

$$S^* = \{ x \in R^* \mid S^*(x) \}.$$

Clearly $S \subseteq S^*$ because $S^*(x)$ agrees with $S(x)$ on R. We shall often write

$$x \in S \text{ instead of } S(x)$$

and

$$x \in S^* \text{ instead of } S^*(x).$$

The upshot of the above is that the Main Theorem also provides for an extension S^* for each $S \subseteq R$ and that we can allow as admissible statements those which involve the sentence fragment $x \in S$; in "lifting" statements from R to R^* we replace the fragment $x \in S$ by $x \in S^*$. Warning! The requirement that admissible statements be permitted only variables ranging over R hasn't been altered. In a given statement the functions, relations, and subsets must remain fixed!

Example 4.1. Let $S = \{ x \in R \mid x < 6 \}$. Now

$$(\forall x)\,(x \in S \leftrightarrow x < 6) \text{ is true in } R$$

so

$$(\forall x)\,(x \in S^* \leftrightarrow x <^* 6) \text{ is true in } R^*.$$

Thus

$$S^* = \{ x \in R^* \mid x <^* 6 \}.$$

Furthermore S^* is a proper extension of S, because for any infinitesimal ε, the number $5 + \varepsilon$ is a member of S^*, but not being a standard number, it can't be a member of S.