5. SUMS OVER INTERVALS OF LENGTH k/5.

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ences for $h(-3p)$ are then consequences of (4.6). The number of summands in $S_{31}(\chi_{4p})$ is $8m + [4j/3]$. If $p \equiv 7 \pmod{12}$, the number of non-zero summands is $4m$; if $p \equiv 11 \pmod{12}$, the number of non-zero summands is $4m + 2$. In either case, the number of non-zero summands is even, and so it follows from (4.6) that $h(-12p) \equiv 0 \pmod 4$ when $p \equiv 3 \pmod 4$. Lastly, the number of summands in $S_{31}(\chi_{8p})$ is $16m + [8j/3]$. If $j = 1$, there are $8m$ non-zero summands; if $j = 5$, there are $8m + 6$ non-zero summands. In either case, $S_{31}(\chi_{8p})$ is even, and we deduce from (4.6) that $h(-24p) \equiv 0 \pmod 4$.

Corollary 4.5. Let p and q be distinct primes with $p, q > 3$ and $p \equiv q \pmod 4$. Then $h(-3pq) \equiv 0 \pmod 4$.

Proof. Let $p = 6m + j$ and $q = 6m' + j'$, where $j, j' = 1$ or 5 and m and m' are non-negative integers. The number of summands in $S_{31}(\chi_{pq})$ is $[pq/3]$, and we observe that $[pq/3] \equiv [jj'/3] \pmod 2$. Of these summands, $[q/3] = 2m' + [j'/3]$ are multiples of p, and $[p/3] = 2m + [j/3]$ are multiples of q. Thus,

$$S_{31}(\chi_{pq}) \equiv [jj'/3] - [j'/3] - [j/3] \pmod 2 .$$

By examining all of the possibilities for the pair j, j', we find that $S_{31}(\chi_{pq})$ is always even. The result now follows from (4.6).

It is clear that the same type of argument yields congruences from $h(-12pq)$ and $h(-24pq)$.

The class number formulae (4.6) and (4.7) appear to be due originally to Lerch [44, pp. 402, 408]. Holden [36] has also given a proof of (4.7).

5. Sums over intervals of length $k/5$.

Theorem 5.1. Let χ be odd and let $\chi_{5k}(n) = \left(\frac{n}{5}\right) \chi(n)$. Then

$$(5.1) \qquad S_{51} = \frac{1}{4\pi i} G(\chi) \{ (5 - \bar{\chi}(5)) L(1, \bar{\chi}) - 5^{1/2} L(1, \bar{\chi}_{5k}) \}$$

and

$$(5.2) \qquad S_{52} = \frac{1}{2\pi i} 5^{1/2} G(\chi) L(1, \bar{\chi}_{5k}) .$$

Proof. Let

$$f(x) = \begin{cases} 1, & 0 < x < 2\pi/5\,, \\ 1/2, & x = 2\pi/5\,, \\ 0, & 2\pi/5 < x \leqslant \pi\,, \end{cases}$$

be an odd function of period 2π. Calculating the Fourier series of f, we find that, for all x,

$$(5.3) \qquad f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin(nx)}{n} - \frac{2}{5\pi}\sum_{n=1}^{\infty}\frac{\sin(5nx)}{n}$$

$$+\frac{2}{\pi}\cos(\pi/5)\sum_{\substack{n=1\\ n\equiv 2,3 \,(\mathrm{mod}\, 5)}}^{\infty}\frac{\sin(nx)}{n} - \frac{2}{\pi}\cos(2\pi/5)\sum_{\substack{n=1\\ n\equiv 1,4 \,(\mathrm{mod}\, 5)}}^{\infty}\frac{\sin(nx)}{n}$$

$$= \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin(nx)}{n} - \frac{2}{5\pi}\sum_{n=1}^{\infty}\frac{\sin(5nx)}{n}$$

$$+\frac{1}{\pi}\cos(\pi/5)\sum_{n=1}^{\infty}\left\{1-\left(\frac{n}{5}\right)\right\}\frac{\sin(nx)}{n}$$

$$-\frac{1}{\pi}\cos(2\pi/5)\sum_{n=1}^{\infty}\left\{1+\left(\frac{n}{5}\right)\right\}\frac{\sin(nx)}{n}$$

$$+\frac{1}{5\pi}\{\cos(2\pi/5)-\cos(\pi/5)\}\sum_{n=1}^{\infty}\frac{\sin(5nx)}{n}$$

$$= \frac{1}{2\pi}\sum_{n=1}^{\infty}\left\{5-5^{1/2}\left(\frac{n}{5}\right)\right\}\frac{\sin(nx)}{n} - \frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{\sin(5n\,x)}{n}\,,$$

since $\cos(\pi/5) = (5^{1/2}+1)/4$ and $\cos(2\pi/5) = (5^{1/2}-1)/4$.

Now, multiply both sides of (2.1) by $\left\{5-5^{1/2}\left(\frac{n}{5}\right)\right\}/(2\pi n)$ and sum on n, $1 \leqq n < \infty$. Next, replace n by $5n$ in (2.1) and then multiply both sides of (2.1) by $-1/(2\pi n)$ and sum on n, $1 \leqq n < \infty$. Adding the resulting two equations and using (5.3), we get

$$2i\,S_{51} = i\sum_{j=1}^{k-1}\chi(j)f(2\pi j/k)$$

$$= \frac{G(\chi)}{2\pi}\left\{\sum_{n=1}^{\infty}\left\{5-5^{1/2}\left(\frac{n}{5}\right)\right\}\frac{\bar{\chi}(n)}{n} - \sum_{n=1}^{\infty}\frac{\bar{\chi}(5n)}{n}\right\}$$

$$= \frac{G(\chi)}{2\pi}\{\,5L(1,\bar{\chi}) - 5^{1/2}L(1,\bar{\chi}_{5k}) - \bar{\chi}(5)\,L(1,\bar{\chi})\,\}\,,$$

from which (5.1) follows immediately.

The proof of (5.2) is similar. In this case, we let

$$f(x) = \begin{cases} 0, & 0 \leqslant x < 2\pi/5,\, 4\pi/5 < x \leqslant \pi, \\ 1/2, & x = 2\pi/5,\, 4\pi/5, \\ 1, & 2\pi/5 < x < 4\pi/5, \end{cases}$$

be an odd function with period 2π. The Fourier series of f is given by

$$f(x) = \frac{5^{1/2}}{\pi} \sum_{n=1}^{\infty} \left(\frac{n}{5}\right) \frac{\sin(nx)}{n} \qquad (-\infty < x < \infty).$$

We then proceed in the same fashion as above.

COROLLARY 5.2. If χ is real and odd, then $S_{52} > 0$.

COROLLARY 5.3. If $d < 0$ and $5 \nmid d$, then

$$S_{51}(\chi_{-d}) = \frac{1}{4}\left\{5 - \left(\frac{d}{5}\right)\right\} h(d) - \frac{1}{4} h(5d) \tag{5.4}$$

and

$$S_{52}(\chi_{-d}) = \frac{1}{2} h(5d)\,. \tag{5.5}$$

Formula (5.5) is due to Lerch [44, p. 407]. By combining (5.4) and (5.5), we can derive a formula for $h(d)$ which is also due to Lerch [44, p. 404].

COROLLARY 5.4. If $p \neq 5$, we have the following consequences:

(5.6) $h(-5p) \equiv 0 \pmod 8$, if $p \equiv 19 \pmod{20}$,

(5.7) $h(-5p) \equiv 4 \pmod 8$, if $p \equiv 11 \pmod{20}$,

(5.8) $h(-5p) \equiv 2h(-p) \pmod 8$, if $p \equiv 7 \pmod{20}$,

(5.9) $h(-5p) \equiv 4 + 2h(-p) \pmod 8$, if $p \equiv 3 \pmod{20}$,

(5.10) $h(-20p) \equiv 0 \pmod 8$, if $p \equiv 1,9 \pmod{20}$ or if $p \equiv 13,37 \pmod{40}$,

(5.11) $h(-20p) \equiv 4 \pmod 8$, if $p \equiv 17,33 \pmod{40}$,

(5.12) $h(-40p) \equiv 4 \pmod 8$, if $p \equiv 2,3 \pmod 5$,

and

(5.13) $h(-40p) \equiv 2h(-8p) \pmod 8$, if $p \equiv 1,4 \pmod 5$.

Proof. If $p \equiv j \pmod{10}$, $1 \leqq j \leqq 9$, then $S_{51}(\chi_p) \equiv [j/5] \pmod 2$. With the use of (5.4) and the above, and recalling that $h(-p)$ is odd, we deduce (5.6)-(5.9).

If $p \equiv j \pmod 5$, $1 \leqq j \leqq 4$, the number of non-zero summands in $S_{51}(\chi_{4p})$ is even if $j = 1$ or 4 and is odd if $j = 2$ or 3. Using also Corollary 3.10, we readily deduce (5.10) and (5.11) from (5.4).

If $p \equiv j \pmod 5$, $1 \leqq j \leqq 4$, the number of non-zero summands in $S_{51}(\chi_{8p})$ is even if $j = 1$ or 4 and is odd if $j = 2$ or 3. Using also the fact that $h(-8p)$ is even, we may deduce (5.12) and (5.13) from (5.4).

COROLLARY 5.5. Let p and q be primes with $p, q \neq 5$ and with $p \equiv q + 2 \pmod 4$. Then

$$h(-5pq) \equiv 0 \pmod 8, \text{ if } p \equiv 1,9 \pmod{20} \text{ and } q \equiv 11,19 \pmod{20},$$

$$h(-5pq) \equiv 4 \pmod 8, \text{ if } p \equiv 13,17 \pmod{20} \text{ and } q \equiv 3,7 \pmod{20},$$

and

$$h(-5pq) \equiv 2h(-pq), \text{ if } p \equiv 1,9 \pmod{20} \text{ and } q \equiv 3,7 \pmod{20},$$
$$\text{or if } p \equiv 13,17 \pmod{20} \text{ and } q = 11,19 \pmod{20}.$$

Of course, the same congruences for $h(-5pq)$ hold if the congruences for p and q are interchanged.

Proof. Let $p \equiv j \pmod{10}$ and $q \equiv j' \pmod{10}$, where $1 \leqq j, j' \leqq 9$. Observe that $S_{51}(\chi_{pq})$ contains $[pq/5]$ terms of which $[q/5]$ are multiples of p and $[p/5]$ are multiples of q. From (5.4), we then find that

$$\begin{aligned} &4([jj'/5] - [j/5] - [j'/5]) \\ &\equiv \left\{5 - \left(\frac{5}{p}\right)\left(\frac{5}{q}\right)\right\} h(-pq) - h(-5pq) \pmod 8 . \end{aligned}$$

Since $h(-pq)$ is even, each of the desired congruences readily follows.

In the case that χ is even, we can state a theorem analogous to Theorem 5.1. However, the L-functions in the representations of S_{51} and S_{52} involve quartic characters. For example,

$$(5.14) \qquad S_{51} = \frac{G(\chi)}{\pi} \left\{ \sin(2\pi/5) \sum_{\substack{n=1 \\ n \equiv 1,4 \pmod 5}}^{\infty} \frac{(-1)^{n+1} \bar{\chi}(n)}{n} + \sin(\pi/5) \sum_{\substack{n=1 \\ n \equiv 2,3 \pmod 5}}^{\infty} \frac{(-1)^{n} \bar{\chi}(n)}{n} \right\};$$

the series on the right side of (5.14) may be written in terms of L-functions of quartic characters. Thus, we are unable to derive any positivity results for character sums.

6. Sums over intervals of length $k/6$.

Theorem 6.1. Let χ be even and let $\chi_{3k}(n) = \left(\frac{n}{3}\right) \chi(n)$. Then

$$S_{61} = \frac{3^{1/2}\, G(\chi)}{2\pi} \{1 + \bar{\chi}(2)\} L(1, \bar{\chi}_{3k}), \tag{6.1}$$

$$S_{62} = -\frac{3^{1/2}\, G(\chi)}{2\pi} \bar{\chi}(2) L(1, \bar{\chi}_{3k}), \tag{6.2}$$

and

$$S_{63} = -\frac{3^{1/2}\, G(\chi)}{2\pi} L(1, \bar{\chi}_{3k}). \tag{6.3}$$

Let χ be odd. Then

$$S_{61} = \frac{G(\chi)}{2\pi i} \{1 + \bar{\chi}(2) + \bar{\chi}(3) - \bar{\chi}(6)\} L(1, \bar{\chi}), \tag{6.4}$$

$$S_{62} = \frac{G(\chi)}{2\pi i} \{2 - \bar{\chi}(2) - 2\bar{\chi}(3) + \bar{\chi}(6)\} L(1, \bar{\chi}), \tag{6.5}$$

and

$$S_{63} = \frac{G(\chi)}{2\pi i} \{1 - 2\bar{\chi}(2) + \bar{\chi}(3)\} L(1, \bar{\chi}). \tag{6.6}$$

We shall not give a proof of Theorem 6.1, because all of the formulas may be deduced from Theorems 3.2 and 4.1 and elementary considerations.

Corollary 6.2. If $d > 0$, we have

$S_{61} > 0$, if d is even, or if $\chi(2) = 1$;

$S_{61} = 0$, if $\chi(2) = -1$;

$S_{62} > 0$, if $\chi(2) = -1$;

$S_{62} = 0$, if d is even;

$S_{62} < 0$, if $\chi(2) = 1$;

$S_{63} < 0$, for all d;