§2. Zeta-functions of Quadratic Fields

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b'_l in place of the b_l, and similarly using the b'_l in (30) would give other weights making (29) hold.

Finally, for completeness' sake we should mention that Siegel gave a somewhat more general formula than the one stated. If A denotes any ideal class of the field K, then restricting the ideals A in the sum (1) to ideals in the class A gives rise to another meromorphic function, denoted $\zeta(s, A)$. This function also takes on rational values at negative odd integers, and Siegel's formula for these rational numbers is identical to (15) except that one must modify the definition of $\sigma_r(\mathfrak{A})$ by only allowing those ideal divisors $\mathfrak{B}$ in (13) that lie in the class A. In the formulation of Siegel's result just given, this can be simply stated

$$\zeta(1-2m, A) = \sum_{\mathfrak{B} \in A} w(\mathfrak{B}) N(\mathfrak{B})^{2m-1}, \tag{32}$$

with the same weights $w(\mathfrak{B})$ as before.

§2. Zeta-functions of Quadratic Fields

We now specialize to quadratic fields. A totally real quadratic field can be written uniquely as $\mathbf{Q}(d^{1/2})$ with $d > 1$ a square-free integer. Then it is easy to check that

$$\begin{aligned} D &= d \quad \text{if} \quad d \equiv 1 \pmod 4, \\ D &= 4d \quad \text{if} \quad d \equiv 2 \text{ or } 3 \pmod 4, \end{aligned} \tag{1}$$

and

$$\mathfrak{d} = (\sqrt{D}), \tag{2}$$

i.e. the different is a principal ideal. The decomposition of rational primes in the ring of integers $\mathcal{O}$ is described in terms of the primitive character $\chi \pmod D$ defined by

$$\chi(x) = \left(\frac{D}{x}\right) \tag{3}$$

(here χ is completely multiplicative, and given on primes by: $\chi(p) = 0$ if $p \mid D$; for $p \nmid 2D$, $\chi(p)$ is ± 1 according as D is or is not a quadratic residue $(\operatorname{mod} p)$; for $p = 2$ and $D = d$ odd, $\chi(2) = (-1)^{(d-1)/4}$) as follows: if $p = 2, 3, 5, \ldots$ is a rational prime, then the ideal $(p) \subset \mathcal{O}$ decomposes into prime ideals according to the value of $\chi(p)$ —

$$\chi(p) = 1 \Rightarrow (p) = \mathfrak{P}_1\mathfrak{P}_2, \quad \mathfrak{P}_1 = \mathfrak{P}_2', \ N(\mathfrak{P}_i) = p, \tag{4a}$$
$$\chi(p) = 0 \Rightarrow (p) = \mathfrak{P}^2, \quad N(\mathfrak{P}) = p, \tag{4b}$$
$$\chi(p) = -1 \Rightarrow (p) = \mathfrak{P}, \quad N(\mathfrak{P}) = p^2. \tag{4c}$$

Substituting this into the Euler product 1 (2) gives (for $Re(s)$ sufficiently large)

$$\begin{aligned}
\zeta_K(s) &= \prod_{\mathfrak{P}} \frac{1}{1 - N(\mathfrak{P})^{-s}} \\
&= \prod_{\chi(p)=1} \frac{1}{1-p^{-s}} \frac{1}{1-p^{-s}} \prod_{\chi(p)=0} \frac{1}{1-p^{-s}} \prod_{\chi(p)=-1} \frac{1}{1-p^{-2s}} \\
&= \prod_{p} \frac{1}{1-p^{-s}} \frac{1}{1-\chi(p)\,p^{-s}} \\
&= \zeta(s)\ L(s,\chi),
\end{aligned} \tag{5}$$

where $\zeta(s)$ is defined in 1 (7) and

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} \tag{6}$$

is the L-series associated to the character χ. Again, (6) is convergent only for $Re(s)$ large enough, but the function $L(s, \chi)$ it defines can be extended to the whole s-plane (and (5) is then true everywhere). $L(s, \chi)$ is holomorphic everywhere.

Since we know the values of $\zeta(2m)$ (equation 1 (9)), we only need calculate $L(2m, \chi)$. But $\chi(n)$ is periodic with period D and satisfies $\chi(n) = \chi(-n)$, so we have

$$L(2m,\chi) = \sum_{n=1}^{\infty} \chi(n)\, n^{-2m} = \frac{1}{2} \sum_{a=1}^{D-1} \chi(a)\, \varphi(a, D; 2m), \tag{7}$$

where

$$\varphi(a, D; 2m) = \sum_{\substack{n=-\infty \\ n \equiv a \ (\mathrm{mod}\ D)}}^{\infty} n^{-2m}. \tag{8}$$

This last sum can be evaluated in terms of elementary functions:

$$\varphi(a, D; 2m) = \sum_{r \in \mathbf{Z}} (rD + a)^{-2m} = D^{-2m} f_m\left(\frac{a}{D}\right), \tag{9}$$

where

$$f_m(x) = \sum_{r \in \mathbf{Z}} \frac{1}{(r+x)^{2m}}$$

$$= \frac{-1}{(2m-1)!} \frac{d^{2m-1}}{dx^{2m-1}} \sum_{-\infty}^{\infty} \frac{1}{r+x}$$

$$= \frac{-\pi}{(2m-1)!} \frac{d^{2m-1}}{dx^{2m-1}} \cot \pi x . \tag{10}$$

Thus

$$f_1(x) = \pi^2 (\csc^2 \pi x), \quad f_2(x) = \pi^4 (\csc^4 \pi x - \frac{2}{3} \csc^2 \pi x). \tag{11}$$

This gives a finite and elementary expression for $L(2m, \chi)$. It can be simplified yet further by observing that $f_m\left(\frac{a}{D}\right)$ is periodic in a with period D, and therefore has a finite Fourier expansion as a sum $\Sigma \gamma_n e^{2\pi i n a/D}$. The coefficients γ_n are easy to compute and are rational. If we then put all this into (7), we finally obtain the formula

$$L(2m, \chi) = \frac{(-1)^{m-1} 2^{2m-1} \pi^{2m}}{(2m-1)! \sqrt{D}} \sum_{j=1}^{D} \chi(j) B_{2m}\left(\frac{i}{D}\right), \tag{12}$$

where $B_r(x)$ denotes the r-th *Bernoulli polynomial:*

$$B_r(x) = \sum_{s=0}^{r} \binom{r}{s} B_{r-s} x^s . \tag{13}$$

If we substitute (12) and 1 (9) into equation (5) and apply the functional equation 1 (6), we obtain finally

$$\zeta_K(1-2m) = \frac{B_{2m}}{4m^2} D^{2m-1} \sum_{j=1}^{D} \chi(j) B_{2m}\left(\frac{i}{D}\right). \tag{14}$$

That is, for quadratic fields it is possible to give a completely elementary formula, derived in a completely elementary way, for the value of $\zeta_K(1-2m)$.

As an illustration, we take $m = 1$. Since

$$B_2(x) = x^2 - x + \frac{1}{6}, \tag{15}$$

one gets (after some trivial manipulations)

$$\zeta_K(-1) = \frac{1}{24D} \sum_{j=1}^{D-1} \chi(j) j^2 . \tag{16}$$

For example, with $K = \mathbf{Q}(\sqrt{5})$ we get

$$\zeta_K(-1) = \frac{1}{120} [1^2 - 2^2 - 3^2 + 4^2] = \frac{1}{30}, \tag{17}$$

while for $K = \mathbf{Q}(\sqrt{13})$

$$\zeta_K(-1)$$

$$= \frac{1}{24 \times 13}[1^2 - 2^2 + 3^2 + 4^2 - 5^2 - 6^2 - 7^2 - 8^2 + 9^2 + 10^2 - 11^2 + 12^2]$$

$$= \frac{1}{6}. \tag{18}$$

For a more complete discussion of the formulas treated in this section, see Siegel [8].

§3. The Siegel Formula for Quadratic Fields

In this section we shall exploit the simple arithmetic of quadratic fields to evaluate in elementary form the various terms entering into Siegel's formula, thus arriving at an expression for $\zeta_K(1-2m)$ which is elementary in the sense that it involves only rational integers and not algebraic numbers or ideals.

We have to evaluate $s_l^K(2m)$, and to do so we must first know how to compute $\sigma_r(\mathfrak{A})$ for any ideal $\mathfrak{A}$.

Lemma. *Let $\mathfrak{A}$ be any ideal of the ring of integers $\mathcal{O}$ of a quadratic field K. Let D be the discriminant of K and $\chi(j) = \left(\frac{D}{j}\right)$ the associated character (as in §2). Then, for any $r \geqslant 0$,*

$$\sigma_r(\mathfrak{A}) = \sum_{j \mid \mathfrak{A}} \chi(j) j^r \sigma_r(N/j^2), \tag{1}$$

where $N = N(\mathfrak{A})$ is the norm of $\mathfrak{A}$, the function σ_r on the right-hand side is the arithmetic function of 1 (12), *and the sum is over all positive integers j dividing $\mathfrak{A}$* (i.e. $\nu/j \in \mathcal{O}$ for every $\nu \in \mathfrak{A}$; clearly this implies $j^2 \mid N$, so equation (1) makes sense).

Proof: It is very easy to check that both sides of (1) are multiplicative functions, i.e. $\sigma_r(\mathfrak{A}\mathfrak{B}) = \sigma_r(\mathfrak{A})\sigma_r(\mathfrak{B})$ for relatively prime ideals $\mathfrak{A}$ and $\mathfrak{B}$, and similarly for the expression on the right-hand side of (1). It therefore suffices to take $\mathfrak{A}$ to be a power $\mathfrak{P}^m$ of a prime ideal $\mathfrak{P}$. Write $N(\mathfrak{P}) = p^i$