

§3. Proof of Theorems 1 and 3

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This follows from Theorem 2: the right-hand side is at most a multiple of the left since $F(x, t) \rightarrow F(x, 0)$ in L_w^p ; the converse inequality is just (3) with w_2 and r chosen to be w and p , resp.

§3. PROOF OF THEOREMS 1 AND 3

We will prove Theorem 1 first, beginning with part (i). Let $F \in H_w^1$, $F = (u, v_1, \dots, v_n)$, $w \in A_1$. By Theorem 2, F has boundary values $F(x, 0) = (f(x), g_1(x), \dots, g_n(x))$ pointwise a.e. and in L_w^1 . In particular, $f, g_1, \dots, g_n \in L_w^1$. We will show that $u = P(f)$ and $v_j = P(g_j)$. Since $u(x, s)$ converges to $f(x)$ in L_w^1 , $P(u(\cdot, s))(x, t) \rightarrow (Pf)(x, t)$ as $s \rightarrow 0$:

$$\begin{aligned} |P(u(\cdot, s))(x, t) - (Pf)(x, t)| &= \left| \int_{R^n} [u(y, s) - f(y)] P(x-y, t) dy \right| \\ &\leq \|u(\cdot, s) - f\|_{1,w} \left\{ \sup_y w(y)^{-1} P(x-y, t) \right\}, \end{aligned}$$

where the expression in curly brackets is finite for each (x, t) (see (6)). By Lemma 1, $u(x, s+t) = P(u(\cdot, s))(x, t)$ since u is harmonic. Hence, letting $s \rightarrow 0$, we obtain $u(x, t) = (Pf)(x, t)$, as desired. The argument proving that $v_j = P(g_j)$ is similar.

Now let $G = (Pf, Q_1 f, \dots, Q_n f)$. Then G is a Cauchy-Riemann system with the same first component as F . This implies that the first component of $F-G$ is zero, and so that the others are independent of t ; that is, $v_j - Q_j f$ is independent of t . Thus, $v_j = Q_j f$ if both $v_j(x, t)$ and $(Q_j f)(x, t)$ tend to zero as $t \rightarrow +\infty$ (x fixed). We have already observed this for $Q_j f$. For v_j , the mean-value property of harmonic functions gives

$$\begin{aligned} |v_j(x, t)| &\leq ct^{-n-1} \iint_{|\xi-x|^2 + |t-\eta|^2 < t^2} |v_j(\xi, \eta)| d\xi d\eta \\ &\leq ct^{-n} \sup_{\eta > 0} \int_{|\xi-x| < t} |v_j(\xi, \eta)| d\xi \\ &\leq ct^{-n} \left(\sup_{\eta > 0} \int_{R^n} |v_j(\xi, \eta)| w(\xi) d\xi \right) \left(\sup_{\xi: |\xi-x| < t} w(\xi)^{-1} \right) \\ &\leq ct^{-n} \sup_{\xi: |\xi-x| < t} w(\xi)^{-1}. \end{aligned}$$

Since $w(\xi)^{-1} \leq c(1+|\xi|)^{n\delta}$ for some $\delta, 0 < \delta < 1$, we have

$$|v_j(s, t)| \leq ct^{-n}(1+|x|+t)^{n\delta}.$$

Hence, $v_j(x, t) \rightarrow 0$ as $t \rightarrow \infty$ for each x .

We now know $u = Pf, v_j = P(g_j) = Q_j f$. Letting $t \rightarrow 0$ in the equation $P(g_j)(x, t) = (Q_j f)(x, t)$ gives $g_j(x) = (R_j f)(x)$ a.e. Thus, $R_j f \in L_w^1$ and $v_j = P(R_j f) = Q_j f$, as desired. All that remains to prove in (i) is that $\|F\|$ and $\|f\|_{1,w} + \sum_{j=1}^n \|R_j f\|_{1,w}$ are equivalent. This, however, follows immediately from (10) with $p = 1$, since

$$F(x, 0) = (f(x), R_1 f(x), \dots, R_n f(x)).$$

To prove (ii), let f be a function in L_w^1 for which each $R_j f \in L_w^1$. (The existence of $R_j f$ as a pointwise limit is guaranteed by the hypothesis $w \in A_1$.) We will show that the vector defined by

$$F = (Pf, Q_1 f, \dots, Q_n f)$$

is in H_w^1 . Once this is done, the rest of (ii) clearly follows from (i). We know F is a Cauchy-Riemann system, and only need to show $\|F\| < +\infty$. As $t \rightarrow 0$, $F(x, t)$ converges a.e. to $(f, R_1 f, \dots, R_n f) = F(x, 0)$, say, so that $|F(x, 0)| \in L_w^1$. Hence, $\|F\| < +\infty$ by Theorem 2 if there exist p and $w_1, \frac{n-1}{n} < p < \infty, w_1 \in A_{pn/(n-1)}$, such that

$$(11) \quad \sup_{t>0} \int_{\mathbb{R}^n} |F(x, t)|^p w_1(x) dx < +\infty.$$

We first claim that if $w \in A_1$, there exists $\alpha > 0$ such that the function

$$w_1(x) = \frac{w(x)}{(1+|x|)^\alpha}$$

also belongs to A_1 . Note that $(1+|x|)^{-\beta} \in A_1$ if $0 \leq \beta < n$, and that there exists $s > 1$ such that $w^s \in A_1$. Hence, for any cube I , Hölder's inequality gives

$$\frac{1}{|I|} \int_I w_1(x) dx \leq \left(\frac{1}{|I|} \int_I w(x)^s dx \right)^{1/s} \left(\frac{1}{|I|} \int_I (1+|x|)^{-\alpha s'} dx \right)^{1/s'},$$

$s' = s/(s-1)$. Choose $\alpha > 0$ so small that $\alpha s' < n$. Then both w^s and $(1+|x|)^{-\alpha s'}$ are in A_1 , and

$$\begin{aligned} \frac{1}{|I|} \int_I w_1(x) dx &\leq c (\text{ess inf}_I w^s)^{1/s} (\text{ess inf}_I (1+|x|)^{-\alpha s'})^{1/s'} \\ &= c (\text{ess inf}_I w) (\text{ess inf}_I (1+|x|)^{-\alpha}) \\ &\leq c \text{ess inf}_I w_1. \end{aligned}$$

This proves the claim.

With this choice of w_1 , we will complete the proof of (ii) by showing that (11) holds for any $p < 1$ which is sufficiently close to 1. Let

$$(R^*f)(x) = \max_{j=1, \dots, n} (R_j^*f)(x).$$

Then, as is well-known, there is a constant c depending only on n such that

$$|F(x, t)| \leq c [f^*(x) + (R^*f)(x)].$$

It follows from the weak-type estimates referred to in §2 that the radial maximal function $N_0(F)(x)$ ($= \sup_{t>0} |F(x, t)|$) satisfies

$$m_w \{x: N_0(F)(x) > \lambda\} \leq c \lambda^{-1} \|f\|_{1,w}, \quad \lambda > 0.$$

We will show that any non-negative function ϕ with

$$m_w \{\phi(x) > \lambda\} \leq c \lambda^{-1}, \quad \lambda > 0,$$

belongs to $L_{w_1}^p$, $1 - \frac{\alpha}{n} < p < 1$. Let $g_r(\lambda)$, $\lambda > 0$, denote the non-increasing rearrangement of a function g with respect to the measure $w(x) dx$. Then, by [5], p. 257,

$$\begin{aligned} \int_{R^n} \phi^p w_1 dx &= \int_{R^n} \phi(x)^p (1+|x|)^{-\alpha} w(x) dx \\ &\leq \int_0^\infty \phi_r^p(\lambda) \{(1+|x|)^{-\alpha}\}_r(\lambda) d\lambda. \end{aligned}$$

We have $\phi_r(\lambda) \leq c \lambda^{-1}$ and must estimate $\{(1+|x|)^{-\alpha}\}_r$. However,

$$m_w \{x: (1+|x|)^{-\alpha} > \lambda\} = m_w \{x: 1+|x| < \lambda^{-1/\alpha}\},$$

which for $\lambda \geq 1$ is zero and for $0 < \lambda < 1$ is less than

$$\int_{|x|<\lambda^{-1/\alpha}} w dx \leq c \lambda^{-n/\alpha} \int_{|x|<1} w dx = c \lambda^{-n/\alpha}$$

(see (5)). Therefore,

$$\{(1+|x|)^{-\alpha}\}_r(\lambda) \leq c(1+\lambda)^{-\alpha/n}, \quad \lambda > 0.$$

Combining estimates, we obtain

$$\int_{R^n} \phi^p w_1 dx \leq c \int_0^\infty \lambda^{-p} (1+\lambda)^{-\alpha/n} d\lambda < +\infty$$

if $1 - \frac{\alpha}{n} < p < 1$, as desired. This completes the proof of (ii).

To prove Theorem 3, let $f \in L_w^1$ and $w \in A_1$. Then (11) holds for F , p and w_1 as in the proof of Theorem 1 (ii). (The proof of (11) does not require $R_j f \in L_w^1$.) Hence, by Lemma 2 (see (8)),

$$N(F)(x) \leq c(|F(x, 0)|^{\frac{n-1}{n}})^{* \frac{n}{n-1}}.$$

Since $F(x, 0) = (f(x), (R_1 f)(x), \dots, (R_n f)(x))$, the conclusion of Theorem 3 follows immediately with $\mu = (n-1)/n$.

To prove the fact stated at the end of the introduction, let

$$f, R_1 f, \dots, R_n f \in L^1.$$

Clearly,

$$\begin{aligned} P(R_j f) \hat{\ } (x, t) &= \hat{P}(x, t)(R_j f) \hat{\ } (x) = e^{-2\pi t|x|} (R_j f) \hat{\ } (x), \\ (Q_j f) \hat{\ } (x, t) &= \hat{Q}_j(x, t)\hat{f}(x) = i \frac{x_j}{|x|} e^{-2\pi t|x|} \hat{f}(x) \text{ a.e.}, \end{aligned}$$

where the Fourier transform is taken in the x variable with t fixed. (Note that for fixed t , $P(x, t)$ belongs to L^1 and $Q_j(x, t)$ belongs to L^2 .) However, these expressions are all equal everywhere since $P(R_j f) = Q_j f$ by Theorem 1 and $P(R_j f) \in L^1$. Therefore, $(R_j f) \hat{\ } (x) = ix_j |x|^{-1} \hat{f}(x)$, as claimed.

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