SUMMABILITY OF SINGULAR VALUES OF L^2 KERNELS. ANALOGIES WITH FOURIER SERIES

Autor(en): **Cochran, James Alan**

Objekttyp: **Article**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **22 (1976)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **24.05.2024**

Persistenter Link: https://doi.org/10.5169/seals-48180

Nutzungsbedingungen

Die ETH-Bibliothek ist Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Inhalten der Zeitschriften. Die Rechte liegen in der Regel bei den Herausgebern.
Die auf der Plattform e-periodica veröffentlichten Dokumente stehen für nicht-kommerzielle Zwecke in Lehre und Forschung sowie für die private Nutzung frei zur Verfügung. Einzelne Dateien oder Ausdrucke aus diesem Angebot können zusammen mit diesen Nutzungsbedingungen und den korrekten Herkunftsbezeichnungen weitergegeben werden.
Das Veröffentlichen von Bildern in Print- und Online-Publikationen ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Die systematische Speicherung von Teilen des elektronischen Angebots auf anderen Servern bedarf ebenfalls des schriftlichen Einverständnisses der Rechteinhaber.

Haftungsausschluss

Alle Angaben erfolgen ohne Gewähr für Vollständigkeit oder Richtigkeit. Es wird keine Haftung übernommen für Schäden durch die Verwendung von Informationen aus diesem Online-Angebot oder durch das Fehlen von Informationen. Dies gilt auch für Inhalte Dritter, die über dieses Angebot zugänglich sind.

Ein Dienst der *ETH-Bibliothek*
ETH Zürich, Rämistrasse 101, 8092 Zürich, Schweiz, www.library.ethz.ch

http://www.e-periodica.ch

SUMMABILITY OF SINGULAR VALUES OF L^2 KERNELS. ANALOGIES WITH FOURIER SERIES

by James Alan COCHRAN

1. INTRODUCTION

The exploitation of analogies between related mathematical contructs is an often-fruitful endeavor. A thorough grounding in finite-dimensional vector spaces enhances the mastery of Hilbert space concepts; knowledge of the characteristic behavior of harmonic functions suggests properties which can be shown to be shared by solutions of far more general elliptic partial differential equations; the convergence question for a given infinite series is made clear through investigation of a related improper integral. Other examples abound, including the reader's own personal favorite.

In this paper we shall be concerned with L^2 kernels, i.e. two-variable functions $K(x, y)$ defined for $a \leqslant x, y \leqslant b$ and satisfying

$$\| K \| \equiv \left[\int_a^b \int_a^b | K(x, y) |^2 \, dx \, dy \right]^{\frac{1}{2}} < \infty,$$

which are envisioned as the kernels of linear Fredholm integral equations. As is customary, we term the nonnegative square roots of the characteristic values of the related kernel $KK^*(x, y)$, the singular values μ_n of the original kernel. Our specific interest is in the connection between the *smoothness* of the given kernel K and the *growth behavior* of these singular values. More particularly, we explore, illuminate, and in general "exploit" the remarkable analogies that prevail between this growth behavior of singular values associated with square-integrable kernels satisfying various smoothness criteria and the values of convergence exponents for classical Fourier series under comparable conditions.

The existence of at least some sort of relationship which permits these analogies is certainly to be suspected in view of the parallelism of the well-known Fourier series result of M. Riesz (see Hardy and Littlewood [20], Bary [1], pp. 184ff, or Zygmund [37], p. 251) that "the Fourier series of an L^2 function f converges absolutely if and only if f can be represented as the convolution of two other L^2 functions," on the one hand, and the nuclear kernel result (Chang [7], [8]; see also Cochran [11], pp. 236-237) that "the series of reciprocal singular values of an L^2 kernel K converges (absolutely, of course) if and only if K can be represented as the composition of two other L^2 kernels," on the other. Especially venturesome readers might even be willing to conjecture such a relationship merely on the basis of the considerable use, over the years, of periodic functions of one variable to generate difference kernels of two variables having specified properties. The carry over of growth/smoothness connections, of course, is immediate in these special cases. Indeed, we need only recall that if $f(x)$, $-\pi \leqslant x \leqslant \pi$, is square-integrable, periodic with period 2π, and has the classical Fourier series coefficients c_n, then the correspondence

$$K(x, y) \equiv f(x-y) \qquad -\pi \leqslant x, y \leqslant \pi$$

leads to a (normal) kernel with singular values

$$\mu_n = 1/2\pi \, | c_n | .$$

We should expect the analysis of the general situation to be considerably more complicated, however.

Perhaps somewhat surprisingly then it actually turns out that the specific relationship which makes possible the general analogies which are the subject of this paper is not an exceedingly deep result, when viewed in the appropriate context, and we shall consider it carefully in a later section. For the present we merely note that the relationship was essential for an investigation carried out by Smithies and reported on already in 1937 [24]. Since the harvest is so rich, we can only conjecture why the relationship lay fallow for so many years and only recently was "rediscovered" and put to full use [13].

In the next section of this paper we list the various classical Fourier series results with which we shall be concerned. These include the several sufficiency conditions for absolute convergence of Fourier series due to Bernstein and Zygmund, for example, as well as numerous more precise results of Hardy and Littlewood, Szász, and others. Subsequently, in Section 3, we gather together the mathematical machinery needed for the investigation of the analogous spectral-theoretic results. A full discussion of the growth behavior of the singular values for the various kernel smoothness conditions of interest is then given in Section 4, along with some additional historical perspective.

2. Fourier Series Results

Let the integrable function $f(x)$, $-\pi \leqslant x \leqslant \pi$, have period 2π, so that $f(x+2\pi) = f(x)$, and in particular $f(\pi) = f(-\pi)$, and assume that $0 < \alpha \leqslant 1$ and $p \geqslant 1$. Denote by Δf one of the three differences (it matters not which for our purposes)

$$f(x) - f(x-h), \quad f(x+h) - f(x), \quad f(x+h) - f(x-h).$$

If $\Delta f = O(|h|^{\alpha})$ we say either that $f(x)$ *belongs to* Lip α or that $f(x)$ *satisfies a Lipschitz condition with exponent* α. More generally, $f(x)$ is said to *belong to the Lipschitz class* Lip (α, p) if

$$\int_{-\pi}^{\pi} |\Delta f|^p \, dx = O(|h|^{\alpha p}).$$

In view of Hölder's inequality, a function of Lip (α, p) belongs to Lip (α, q) for all $1 \leqslant q < p$. Moreover, a function of Lip α clearly belongs to Lip (α, p) for all $p \geqslant 1$. In fact, the class Lip α may be viewed roughly as the limit of Lip (α, p) for $p = \infty$.

The classical complex Fourier series of $f(x)$ is defined by

$$f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{inx} \text{ where } c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} \, dx.$$

Equivalently, if $c_n \equiv \frac{1}{2}(a_n - ib_n)$ for all n, then

$$f(x) \sim \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

with

$$\begin{matrix} a_n \\ b_n \end{matrix} = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \begin{matrix} \cos \\ \sin \end{matrix} nx \, dx.$$

For given integrable f, the series

$$\sum_{n=-\infty}^{\infty} |c_n|^{\gamma}$$

of moduli of the coefficients of these Fourier series may not converge for any finite $\gamma > 0$. If it does for certain γ, however, the convergence exponent

ρ of the Fourier coefficients is the infimum of these γ. For square-integrable f, we know that $\rho \leqslant 2$. (The above series, of course, need not be convergent for $\gamma = \rho$.)

The earliest result of interest to us here is the well-known theorem of Bernstein [2], [3], [4] (see also Bary [1], pp. 153-171, or Zygmund [37], pp. 240-243, for example) which we state as follows:

THEOREM 2.1. *If* $f(x)$ *is in* Lip α *with* $\alpha > \frac{1}{2}$, *then* $\rho < 1$.

This result has a sharpened form due to Szász [26], namely:

THEOREM 2.2. *If* $f(x)$ *is in* Lip α, *then* $\rho = 1/(\alpha + 1/2)$,

and an even more general rendition due essentially to Szász [26] (the case $p = 2$), [27], Titchmarsh [28] (the corresponding theorem for transforms; see also [29]), and Hardy and Littlewood [19] (under the assumption $\alpha p > 1$):

THEOREM 2.3. *If* $f(x)$ *belongs to* Lip (α, p), *then*

$$\rho = \begin{cases} \dfrac{1}{\alpha + 1 - 1/p} & 1 \leqslant p \leqslant 2 \\[2ex] \dfrac{1}{\alpha + 1/2} & p > 2. \end{cases}$$

For square-integrable f, this result only has relevance, of course, when $2\alpha p > 2 - p$.

We note in passing that since the class Lip $(1, p)$, where $p > 1$, is equivalent to the collection of integrals of functions of the Lebesgue class L^p (Hardy and Littlewood [18], p. 599), Theorem 2.3 has as a special case the well-known result originally established by Tonelli [30]:

COROLLARY. *If* $f(x)$ *is absolutely continuous and its derivative* $f'(x)$ *belongs to* $L^p, p > 1$, *then* $\rho < 1$.

Other restrictions on $f(x)$, $-\pi \leqslant x \leqslant \pi$, are also of interest to us. Finite-valued functions are said to be of *bounded variation* if for all $N \geqslant 1$ and arbitrary choice of partition $-\pi \leqslant x_0 \leqslant x_1 \leqslant \ldots \leqslant x_N \leqslant \pi$,

$$\sum_{n=1}^{N} |f(x_n) - f(x_{n-1})| \leqslant B \text{ (const.)} < \infty .$$

Since $f(x)$ is in Lip $(1, 1)$ if and only if (iff) it is of bounded variation, no new results arise without at least some modest additional assumptions beyond mere bounded variation. One such set of combined restrictions leads to the following classical result first established by Zygmund [35] (see also Bary [1], Zygmund [37]):

THEOREM 2.4. *If* $f(x)$ *is of bounded variation and also in* Lip β *for some* $\beta > 0$, *then* $\rho < 1$.

Here also there is a sharpened form, this time due to Waraszkiewicz [31] (see also Zygmund [36]):

THEOREM 2.5. *If* $f(x)$ *is of bounded variation and also in* Lip β *for some* $\beta > 0$, *then* $\rho = 1/(1+\beta/2)$.

Other results, employing different sets of combined assumptions, can be established using the convexity property of the class Lip (α, p) (Hardy and Littlewood [20]), namely:

PROPERTY 1. If $f(x)$ belongs both to Lip (α, p) and to Lip (β, q), where $p < q$, then it belongs to Lip (γ, r) for all $p \leqslant r \leqslant q$, where

$$\gamma = \alpha \frac{p(q-r)}{r(q-p)} + \beta \frac{q(r-p)}{r(q-p)} .$$

In the limiting case $q = \infty$, where $f(x)$ is in Lip β, then

$$\gamma = \beta + (\alpha - \beta) \frac{p}{r} .$$

Interplaying this property with the earlier Theorem 2.3, we obtain the general

THEOREM 2.6. *If* $f(x)$ *belongs both to* Lip (α, p) *and to* Lip (β, q), *where* $p < q$, *then*

i) *for* $q \leqslant 2$,

$$\rho = \begin{cases} \dfrac{1}{\alpha + 1 - 1/p} & pq(\alpha-\beta) > q - p \\[2ex] \dfrac{1}{\beta + 1 - 1/q} & pq(\alpha-\beta) \leqslant q - p , \end{cases}$$

ii) *for* $p \leqslant 2 < q$,

$$\rho = \begin{cases} \dfrac{1}{\alpha + 1 - 1/p} & pq(\alpha - \beta) > q - p \\[2ex] \dfrac{2(q-p)}{q(2\beta + \alpha p + 1) - p(2\alpha + \beta q + 1)} & 0 < pq(\alpha - \beta) \leqslant q - p \\[2ex] \dfrac{1}{\beta + 1/2} & \alpha \leqslant \beta, \end{cases}$$

iii) *and for* $p > 2$,

$$\rho = \begin{cases} \dfrac{1}{\alpha + 1/2} & \alpha > \beta \\[2ex] \dfrac{1}{\beta + 1/2} & \alpha \leqslant \beta. \end{cases}$$

Theorem 2.5 is the special case of this result when $\alpha = p = 1$, $q = \infty$. Other special cases are:

Corollary 1. *If* $f(x)$ *is of bounded variation and also in* Lip (β, q) *for some* $\beta > 0$, $q \geqslant 1$, *then*

$$\rho = \begin{cases} 1 & \beta q < 1 \\[2ex] \dfrac{q}{\beta q + q - 1} & \beta q \geqslant 1, q \leqslant 2 \\[2ex] \dfrac{2(q-1)}{\beta q + 2q - 3} & \beta q \geqslant 1, q > 2; \end{cases}$$

Corollary 2. *If* $f(x)$ *belongs to* Lip (α, p) *and also satisfies an ordinary Lipschitz condition with exponent* $\beta > 0$, *then*

$$\rho = \begin{cases} \dfrac{p}{\alpha p + p - 1} & p(\alpha - \beta) > 1, \quad p \leqslant 2 \\[2ex] \dfrac{2}{\beta(2-p) + \alpha p + 1} & 0 < p(\alpha - \beta) \leqslant 1, \quad p \leqslant 2 \\[2ex] \dfrac{1}{\alpha + 1/2} & \alpha > \beta, \quad p > 2 \\[2ex] \dfrac{1}{\beta + 1/2} & \alpha \leqslant \beta. \end{cases}$$

We note that $\rho < 1$ for $\beta q > 1$ in the first case, while for $p \leqslant 2$, $\alpha > \beta > (1-\alpha p)/(2-p)$ gives the same conclusion in the latter situation. Comparable results were observed by Hardy and Littlewood [20] and Waraszkiewicz [31].

Perhaps not surprisingly, the Corollary to Theorem 2.3 may be viewed as a special case of Corollary 2 above since when $\alpha p > 1$, functions in $\mathrm{Lip}\,(\alpha, p)$ likewise belong to $\mathrm{Lip}\,(\alpha - 1/p + 1/q, q)$ for all $q > p$ and hence are equivalent to functions in $\mathrm{Lip}\,(\alpha - 1/p)$ (Hardy and Littlewood [19]). Alternatively, the earlier result can also be established using the following variant of one-half of the Hausdorff-Young Theorem (Hausdorff [21], Young [33], [34]; see also Hardy and Littlewood [17], Bary [1], Zygmund [37]) and the familiar relation between the Fourier coefficients of $f(x)$ and its derivatives $f^{(s)}(x)$, $s = 1, 2, \ldots$:

THEOREM 2.7. *If* $f(x)$ *is in* L^p, $p > 1$, *then*

$$\rho = \begin{cases} \dfrac{p}{p-1} & p \leqslant 2 \\ 2 & p > 2. \end{cases}$$

PROPERTY 2. If $f^{(s-1)}(x)$ is absolutely continuous for some positive integer s, then the Fourier coefficients c_{ns} of $f^{(s)}(x)$ are given by

$$c_{ns} = (in)^s c_n .$$

(Here, of course, we have made the tacit assumption that the periodic $f^{(r)}(x)$, $0 \leqslant r \leqslant s-1$, are all continuous in the wide-sense, i.e. for all x, so that in particular $f^{(r)}(\pi) = f^{(r)}(-\pi)$, $0 \leqslant r \leqslant s-1$.) Property 2 easily leads to

PROPERTY 3. If $f^{(s-1)}(x)$ is absolutely continuous for some positive integer s, and the convergence exponent of the Fourier coefficients of $f^{(s)}(x)$ is ρ_s, then

$$\rho = \frac{\rho_s}{1 + s\rho_s} .$$

Taken together, the above results finally yield the general

THEOREM 2.8. *If* $f^{(s-1)}(x)$ *is absolutely continuous for some positive integer* s, *and* $f^{(s)}(x)$ *belongs to* L^p, $p > 1$, *then*

$$\rho = \begin{cases} \dfrac{p}{p(s+1) - 1} & p \leqslant 2 \\[2ex] \dfrac{2}{1 + 2s} & p > 2. \end{cases}$$

In particular, for $s = 1$

$$\rho = \begin{cases} \dfrac{p}{2p - 1} & p \leqslant 2 \\[2ex] \dfrac{2}{3} & p > 2. \end{cases}$$

Any number of other deductions can be obtained by combining Theorem 2.8 with earlier results. We content ourselves with

THEOREM 2.9. *If* $f^{(s-1)}(x)$ *is absolutely continuous for some positive* s, *and if* $f^{(s)}(x)$ *is of bounded variation and also in* Lip (β, q) *for some* $\beta > 0$, $q \geqslant 1$, *then*

$$\rho = \begin{cases} \dfrac{1}{1 + s} & \beta q < 1 \\[2ex] \dfrac{q}{q(\beta + 1 + s) - 1} & \beta q \geqslant 1, \ q \leqslant 2 \\[2ex] \dfrac{2(q-1)}{q(\beta + 2 + 2s) - 3 - 2s} & \beta q \geqslant 1, \ q > 2; \end{cases}$$

THEOREM 2.10. *If* $f^{(s-1)}(x)$ *is absolutely continuous for some positive* s, *and if* $f^{(s)}(x)$ *belongs to* Lip (α, p) *and also satisfies an ordinary Lipschitz condition with exponent* $\beta > 0$, *then*

$$\rho = \begin{cases} \dfrac{p}{p(\alpha + 1 + s) - 1} & p(\alpha - \beta) > 1, p \leqslant 2 \\[2ex] \dfrac{2}{\beta(2-p) + \alpha p + 1 + 2s} & 0 < p(\alpha - \beta) \leqslant 1, p \leqslant 2 \\[2ex] \dfrac{1}{\alpha + s + 1/2} & \alpha > \beta, p > 2 \\[2ex] \dfrac{1}{\beta + s + 1/2} & \alpha \leqslant \beta. \end{cases}$$

3. Preliminaries

Since in this paper we are concerned for the most part with square-integrable kernels $K(x, y)$, and the Lebesgue integral is employed throughout, equalities and inequalities between functions, therefore, are generally to be understood as holding "almost everywhere" (a.e.).

For convenience we take $a = 0$, $b = \pi$, which we may do without any loss of generality, and consider the class of L^2 kernels $K(x, y)$ with $0 \leqslant x, y \leqslant \pi$. In our later work we will need to direct our attention primarily to one of the two variables x, y. Let us choose this to be the first and extend K to be periodic in this variable. There are many ways, of course, to accomplish this task, but a not unreasonable procedure is to first define

$$K^{(r)}(x, y) \equiv \frac{\partial^r K(x, y)}{\partial x^r} \quad (r = 0, 1, \ldots, s) \tag{3.1}$$

for some preselected nonnegative integer s, and then assume that $K(x, y)$ is extended, as an even function of x if s is even, and as an odd function of x if s is odd, into the domain $-\pi \leqslant x \leqslant 0$, and thence as a periodic function of x with period 2π. This approach ensures that, under suitable restrictions, the classical Fourier series for $K^{(s)}(x, y)$, viewed as a function of its first variable, consists solely of sine terms.

In order to enhance the character of the analogies in which we are interested, we shall say that $K^{(s)}(x, y)$ *is in* Lip α if

$$| K^{(s)}(x+h, y) - K^{(s)}(x-h, y) | < | h |^{\alpha} A(y) \quad (0 < \alpha \leqslant 1)$$

where $A(y)$ is nonnegative and square-integrable. More generally, for $p \geqslant 1$, $K^{(s)}(x, y)$ *will be said to be in* Lip (α, p) if

$$\int_0^{\pi} | K^{(s)}(x+h, y) - K^{(s)}(x-h, y) |^p \, dx < | h |^{\alpha p} A^p(y) \quad (0 < \alpha \leqslant 1)$$

with L^2 $A \geqslant 0$. In similar fashion, $K^{(s)}(x, y)$ *will be said to be relatively uniformly of bounded variation* if for all $N \geqslant 1$ and arbitrary choice of partition $0 \leqslant x_0 \leqslant x_1 \leqslant \ldots \leqslant x_N \leqslant \pi$,

$$\sum_{n=1}^{N} | K^{(s)}(x_n, y) - K^{(s)}(x_{n-1}, y) | < B(y)$$

where L^2 $B \geqslant 0$. The comparable definitions appropriate whenever the roles of x and y are reversed should be obvious.

Two-variable kernels behave very much like their one-variable analogues as regards integrated Lipschitz conditions. Indeed, the following can be easily established:

PROPERTY 4. Kernels in Lip (α, p) also belong to Lip (α, q) for all $1 \leqslant q < p$. Kernels in Lip α are automatically in Lip (α, p) for all $p \geqslant 1$.

PROPERTY 5. Kernels which are relatively uniformly of bounded variation belong to Lip (1, 1).

PROPERTY 6. If $K(x, y)$ is absolutely continuous in x, for almost all y, and

$$\int_0^\pi \left[\int_0^\pi | K^{(1)}(x, y) |^p \, dx \right]^{2/p} dy < \infty,$$

$p > 1$, then $K(x, y)$ is in Lip $(1, p)$.

PROPERTY 7. If a kernel belongs both to Lip (α, p) and to Lip (β, q) with $1 \leqslant p < q$, then it belongs to Lip (γ, r) for all $p \leqslant r \leqslant q$, where

$$\gamma = \alpha \frac{p(q-r)}{r(q-p)} + \beta \frac{q(r-p)}{r(q-p)}.$$

A somewhat deeper result is

PROPERTY 8. Whenever $1 \leqslant p \leqslant q$, $pq(\alpha-\beta) \geqslant q - p$, kernels in Lip (α, p) are automatically also in Lip (β, q).

4. Growth Estimates for Singular Values

We come now to the main thrust of our narrative. The characteristic values associated with a given L^2 kernel $K(x, y)$, $0 \leqslant x, y \leqslant \pi$, are those special values of λ for which there exist nontrivial solutions of the homogeneous Fredholm integral equation

$$\phi(x) = \lambda \int_0^\pi K(x, y) \phi(y) \, dy.$$

The singular values are those positive values μ for which there exist nontrivial $\phi(x)$, $\Psi(x)$ satisfying the coupled equations

$$\phi(x) = \mu \int_0^\pi K(x,y)\,\Psi(y)\,dy, \tag{4.1}$$
$$\Psi(x) = \mu \int_0^\pi \overline{K(y,x)}\,\phi(y)\,dy.$$

There are at most countably many of each, and they are customarily ordered (indexed) according to increasing modulus, namely

$$0 < |\lambda_1| \leqslant |\lambda_2| \leqslant \dots$$
$$0 < \mu_1 \leqslant \mu_2 \leqslant \dots .$$

The important inequalities (Weyl [32], Chang [8]; see also Gohberg and Krein [16], p. 41, Cochran [11], pp. 243-245)

$$\sum_{n=1}^{N} \left|\frac{1}{\lambda_n}\right|^p \leqslant \sum_{n=1}^{N} \left(\frac{1}{\mu_n}\right)^p \quad p > 0, N \text{ arbitrary}, \tag{4.2}$$

relate their growth behavior.

The earliest known growth estimates concern characteristic values. In 1909, Schur [23] established for continuous kernels that

$$\sum_n \left|\frac{1}{\lambda_n}\right|^2 \leqslant \|K\|^2$$

(This was subsequently extended to L^2 kernels by Carleman [6]). Even earlier, however, Fredholm himself [15] (see also Cochran [10], [11], pp. 251ff.) had essentially shown that

THEOREM 4.1. *If* $K(x,y)$ *is in* Lip α *with* $\alpha > 1/2$, *then*

$$\sum_n \left|\frac{1}{\lambda_n}\right| < \infty .$$

It is interesting to note that this result which, for characteristic values, mirrors Theorem 2.1, actually predated the work of Bernstein.

Numerous other growth estimates, many of them analogous to our earlier Fourier series results of Section 2, have been established by various investigators. Notable among these are the substantial contributions of Hille and Tamarkin [22] (see also Cochran [11], pp. 251-265). For the most part, though, these pertain to characteristic values, and, in view of the

Weyl-Chang inequalities (4.2), the growth behavior of the singular values is of greater intrinsic interest.

With regards to these singular values, we do know that the associated singular functions given by (4.1) can be chosen to be orthonormal amongst themselves as well as biorthogonal with respect to the kernel K. It then follows that

$$K(x, y) = \sum_n \frac{\phi_n(x)\,\overline{\Psi}_n(y)}{\mu_n} \tag{4.3}$$

where the right-hand side converges in the mean. Moreover,

$$\sum_n \left(\frac{1}{\mu_n}\right)^2 = \| K \|^2, \tag{4.4}$$

so that for L^2 kernels we readily conclude that the series of reciprocal singular values is γ-summable at least for all $\gamma \geqslant 2$. The convergence of $\sum (1/\mu_n)^\gamma$ for exponents γ smaller than 2, however, cannot be established without additional restrictions on the kernel K. [1])

The additional restrictions in which we are interested are of the "smoothness" variety. Let us assume that the square-integrable kernel $K(x, y)$ is also such that the $K^{(r)}(x, y)$ $0 \leqslant r \leqslant s - 2$, (defined by (3.1)), are continuous in x, a.e. in y, for some positive (nonnegative) integer s, $K^{(s-1)}(x, y)$ is absolutely continuous in x, a.e. in y, and $K^{(s)}(x, y)$ is in $L^p(x)$, a.e. in y, for some $p > 1$. Under these conditions Smithies [24] essentially showed that

THEOREM 4.2. *If* $K^{(s)}(x, y)$ *belongs to* $\operatorname{Lip}(\alpha, p)$, *then* $\sum (1/\mu_n)^\gamma$ *converges for all* $\gamma > \rho$ *where*

$$\rho = \begin{cases} \dfrac{1}{\alpha + s + 1 - 1/p} & 1 < p \leqslant 2 \\[2ex] \dfrac{1}{\alpha + s + 1/2} & p > 2. \end{cases}$$

When $s = 0$, the additional proviso $\alpha + 1/2 > 1/p$ may be needed since $K \in L^2$.

---

1) Although now-a-days it is rather routine to convince yourself of this fact (recall our earlier discussion on difference kernels) Carleman [5] was probably the first to carefully establish that even continuity of the kernel was not generally sufficient to ensure convergence for any $\gamma < 2$.

The Smithies proof is very instructive. As a key ingredient it makes use of the fact that the best mean square approximation to a given square-integrable kernel K by degenerate kernels of the form

$$K_N(x,y) = \sum_{n=1}^{N} a_n(x)\,\bar{b}_n(y) \quad a_n, b_n \in L^2 \tag{4.5}$$

occurs, for fixed N, when the a_n, b_n are proportional to the singular functions ϕ_n, Ψ_n of K [25]. Indeed, if we carry out the details we find

$$\begin{aligned} \left\| K(x,y) - \sum_{n=1}^{N} a_n(x)\,\bar{b}_n(y) \right\|^2 &\leqslant \left\| K(x,y) - \sum_{n=1}^{N} \frac{\phi_n(x)\,\overline{\Psi}_n(y)}{\mu_n} \right\|^2 \\ &= \| K \|^2 - \sum_{n=1}^{N} \left(\frac{1}{\mu_n}\right)^2 \\ &= \sum_{n=N+1}^{\infty} \left(\frac{1}{\mu_n}\right)^2, \end{aligned} \tag{4.6}$$

where we have assumed that the singular functions are orthonormalized and then employed (4.4). In the special case, moreover, when the a_n are the appropriate normalized trigonometric functions, namely $\{\sqrt{2/\pi}\cos nx\}$ if s is even and $\{\sqrt{2/\pi}\sin nx\}$ if s is odd (recall the earlier discussion of Section 3 where we imbued K with certain periodicity properties), and the $\bar{b}_n$ are the resulting Fourier coefficients of $K(x,y)$ viewed as a function of x alone, (4.6) takes the form

$$\sum_{n=N+1}^{\infty} \left(\frac{1}{\mu_n}\right)^2 \leqslant \| K \|^2 - \sum_{n=1}^{N} \int_0^{\pi} | b_n(y) |^2 \, dy.$$

In fact, using essentially Parseval's relation, the right-hand side of this inequality can be further rewritten as

$$\sum_{n=N+1}^{\infty} \left(\frac{1}{\mu_n}\right)^2 \leqslant \sum_{n=N+1}^{\infty} \int_0^{\pi} | b_n(y) |^2 \, dy. \tag{4.7}$$

The intimate relationship that exists between the growth behavior of the singular values associated with two-variable kernels and the asymptotic character of allied classical one-variable Fourier coefficients is rather clearly exhibited by the expression (4.7). This, then, is the essential relationship which engenders the desired analogies. Care must be taken in carrying out the details, however, to ensure that each of the $K^{(r)}$, $0 \leqslant r \leqslant s-1$, is continuous in the wide-sense, and thus some modification of the behavior of the $K^{(r)}(x, y)$ for $x = 0, \pi$ may be necessary. Fortunately, this can be accomplished with a degenerate perturbation which, as the following lemma shows, leaves unchanged the fundamental asymptotics in question.

LEMMA.[1]) Let $K(x, y), L(x, y), a \leqslant x, y \leqslant b$, be two L^2 kernels which differ by a degenerate kernel, i.e. $K = L + K_N$ where K_N has the form (4.5) for some fixed positive integer N. Then their respective singular values $\mu_n(K)$, $\mu_n(L)$ satisfy

$$\mu_{n-N}(L) \leqslant \mu_n(K) \leqslant \mu_{n+N}(L)$$

for all $n > N$, and hence

$$\mu_n(K) = O(n^{\gamma}) \text{ iff } \mu_n(L) = O(n^{\gamma}).$$

Returning to Theorem 4.2, although Smithies didn't use the fact, we note that the special case $s = 0$ is the precise analogue of the Fourier series result Theorem 2.3. In view of Property 4, this case also contains the analogues of the earlier Theorems 2.1, 2.2. Recalling Property 6, moreover, the general case of Theorem 4.2 clearly is analogous to Theorem 2.8 and, as such, actually generalizes to arbitrary $p > 1$ a result alternatively established for $p = 2$ by Smithies' student Chang [9] (see also Gohberg and Krein [16], pp. 119-123).

As in the Fourier series situation, the convexity of the class Lip (α, p) plays an important and extremely useful role. Blending Property 7 with Theorem 4.2, for example, we obtain the following extended analogy to Theorem 2.6:

THEOREM 4.3. *If* $K^{(s)}(x, y)$ *belongs both to* Lip (α, p) *and to* Lip (β, q), *with* $p < q$, *then* $\sum (1/\mu_n)^{\gamma}$ *converges for all* $\gamma > \rho$ *where* ρ *is as given in Theorem 2.6 but with* α, β *replaced by* $\alpha + s, \beta + s$ *respectively.*

In fashion similar to before, Properties 4, 5 then lead to the special cases

THEOREM 4.4. *If* $K^{(s)}(x, y)$ *is relatively uniformly of bounded variation and also in* Lip (β, q) *for some* $\beta > 0, q \geqslant 1$, *then* $\sum (1/\mu_n)^{\gamma}$ *converges for all* $\gamma > \rho$ *where* ρ *is as given in Theorem 2.9;*

1) This particular Lemma is a special case of results of Fan [14]. Already in [24], however, Smithies essentially had established the asymptotic invariance property of the singular values under such perturbations.

THEOREM 4.5. *If* $K^{(s)}(x, y)$ *belongs both to* Lip (α, p) *and to* Lip β, *then* $\sum (1/\mu_n)^\gamma$ *converges for all* $\gamma > \rho$ *where* ρ *is as given in Theorem 2.10.*

Naturally, these theorems also contain the analogues of the Zygmund and Waraszkiewicz results, Theorems 2.4, 2.5.

In closing it is worth remarking that all of the above kernel function results are equally as sharp as the corresponding Fourier series results since, as we have seen earlier, for periodic difference kernels the singular values and the related Fourier coefficients are essentially reciprocals. In view of the Weyl-Chang inequalities (4.2), moreover, these theorems amplify and extend our knowledge concerning the growth behavior of the *characteristic* values of "smooth" kernels (see [22], [11], for example).

5. ACKNOWLEDGEMENT

Our original interest in this entire inquiry owed much to numerous stimulating discussions with our former colleague D. W. Swann.

REFERENCES

[1] BARY, N. K. *A treatise on trigonometric series.* Vol. II (translated from the 1961 Russian ed. by M. F. Mullins). Pergamon, New York, 1964.

[2] BERNSTEIN, S. N. Sur la convergence absolue des séries trigonométriques. *C.R. Acad. Sci. Paris 158* (1914), pp. 1661-1663.

[3] —— On the absolute convergence of trigonometric series. *Soobshch. Khar'kov. Mat. Obshch. (2) 14* (1914), pp. 139-144 (in Russian); *ibid.* (1915) pp. 200-201; see also *Collected Works*, Vol. I, 1952, pp. 217-223 (in Russian).

[4] —— Sur la convergence absolue des séries trigonométriques. *C.R. Acad. Sci. Paris 199* (1934), pp. 397-400.

[5] CARLEMAN, T. Über die Fourierkoeffizienten einer stetigen Funktion. *Acta Math. 41* (1918), pp. 377-384; see also *Edition complète des articles*, Malmö, 1960, pp. 15-22.

[6] —— Zur Theorie der linearen Integralgleichungen. *Math. Zeit. 9* (1921), pp. 196-217; see also *Edition complète des articles,* Malmö 1960, pp. 79-100.

[7] CHANG, S. H. A generalization of a theorem of Lalesco. *J. London Math. Soc. 22* (1947), pp. 185-189.

[8] —— On the distribution of the characteristic values and singular values of linear integral equations. *Trans. Amer. Math. Soc. 67* (1949), pp. 351-367. MR 11, 523.

[9] —— A generalization of a theorem of Hille and Tamarkin with applications. *Proc. London Math. Soc. (3) 2* (1952), pp. 22-29. MR 13, 950.

[10] COCHRAN, J. A. The existence of eigenvalues for the integral equations of laser theory. *Bell Syst. Tech. J. 44* (1965), pp. 77-88. MR 30. 1368.

[11] —— *The analysis of linear integral equations.* McGraw-Hill, New York, 1972.

[12] —— The nuclearity of operators generated by Hölder continuous kernels. *Proc. Cambridge Phil. Soc. 75* (1974), pp. 351-356.

[13] —— Growth estimates for the singular values of square-integrable kernels. *Pacific J. Math. 56* (1975), pp. 51-58.

[14] FAN, K. Maximum properties and inequalities for the eigenvalues of completely continuous operators. *Proc. Nat. Acad. Sci. U.S.A. 37* (1951), pp. 760-766.

[15] FREDHOLM, I. Sur une classe d'équations fonctionnelles. *Acta Math. 27* (1903), pp. 365-390; see also *Oeuvres complètes*, Malmö, 1955, pp. 81-106.

[16] GOHBERG, I. C. and M. G. KREIN. Introduction to the theory of linear nonselfadjoint operators. *Translations of Math. Mono., vol. 18, Amer. Math. Soc.*, Providence, R.I., 1969.

[17] HARDY, G. H. and J. E. LITTLEWOOD. Some new properties of Fourier constants. *Math. Ann. 97* (1926), pp. 159-209; see also *Collected papers of G. H. Hardy*, vol. III, Clarendon, Oxford, 1969, pp. 348-399.

[18] —— Some properties of fractional integrals. I. *Math. Zeit. 27* (1928), pp. 565-606; see also *Collected papers of G. H. Hardy, op. cit.*, 1969, pp. 564-607.

[19] —— A convergence criterion for Fourier series. *Math. Zeit 28* (1928), pp. 612-634; see also *Collected papers of G. H. Hardy, op. cit.*, 1969, pp. 28-51.

[20] —— Notes on the theory of series (IX): On the absolute convergence of Fourier series. *J. London Math. Soc. 3* (1928), pp. 250-253; see also *Collected papers of G. H. Hardy, op. cit.*, 1969, pp. 52-56.

[21] HAUSDORFF, F. Eine Ausdehnung des Parsevalschen Satzes über Fourierreihen. *Math. Zeit. 16* (1923), pp. 163-169.

[22] HILLE, E. and J. D. TAMARKIN. On the characteristic values of linear integral equations. *Acta Math. 57* (1931), pp. 1-76.

[23] SCHUR, I. Über die charakteristischen Wurzeln einer linearen Substitution mit einer Anwendung auf die Theorie der Integralgleichungen. *Math. Ann. 66* (1909), pp. 488-510.

[24] SMITHIES, F. The eigen-values and singular values of integral equations. *Proc. London Math. Soc. (2) 43* (1937), pp. 255-279.

[25] —— *Integral equations.* Cambridge, London, 1962, pp. 148-150.

[26] SZÁSZ, O. Über den Konvergenzexponenten der Fourierschen Reihen gewisser Funktionenklassen. *Sitzungsber. Akad. Wiss. München Math. Phys. Kl.* (1922), pp. 135-150; see also *Collected mathematical papers*, Dept. of Math., Cincinnati, Cincinnati, Ohio, 1955, pp. 684-699.

[27] —— Über die Fourierschen Reihen gewisser Funktionenklassen. *Math. Ann. 100* (1928), pp. 530-536; see also *Collected mathematical papers, op. cit.*, 1955, pp. 758-764.

[28] TITCHMARSH, E. C. A note on Fourier transforms. *J. London Math. Soc. 2* (1927), pp. 148-150.

[29] —— *Introduction to the theory of Fourier integrals.* 2nd. ed., Clarendon, Oxford, 1948, pp. 115-117.

[30] TONELLI, L. Sulla convergenza assoluta delle serie di Fourier. *Rend. Accad. Naz. Lincei (6) 2* (1925), pp. 145-149; see also *Opera scelte*, vol. IV, Cremonese, Rome, 1963, pp. 11-16.

[31] WARASZKIEWICZ, Z. Remarque sur un théorème de M. Zygmund. *Pol. Akad. Umiej. Krakow Wydz. Mat.-Przy. A.* (1929), pp. 275-279.

[32] WEYL, H. Inequalities between the two kinds of eigenvalues of a linear transformation. *Proc. Nat. Acad. Sci. U.S.A. 35* (1949), pp. 408-411; see also *Gesammelte abhandlungen*, vol. IV, Springer Verlag, Berlin, 1968, pp. 390-393. MR 11, 37.

[33] YOUNG, W. H. On the multiplication of successions of Fourier constants. *Proc. Roy. Soc. A, 87* (1912), pp. 331-339.

[34] —— On the determination of the summability of a function by means of its Fourier constants. *Proc. London Math. Soc. (2) 12* (1913), pp. 71-88.

[35] ZYGMUND, A. Remarque sur la convergence absolue des séries de Fourier. *J. London Math. Soc. 3* (1928), pp. 194-196.

[36] —— Some points in the theory of trigonometric and power series. *Trans. Amer. Math. Soc. 36* (1934), pp. 586-617.

[37] —— *Trigonometric series.* vols. I, II, University Press, Cambridge, 1968.

(Reçu le 15 janvier 1976)

James Alan Cochran

Department of Mathematics
Virginia Polytechnic Institute and State University
Blacksburg, Virginia 24061

vide-leer-empty

ERRATA

LATTICE POINTS INSIDE A CONVEX BODY

by G. D. Chakerian

(*L'Enseignement Mathématique 20* (1974), pp. 243-245).

It has been brought to my attention that the main theorem in my note is not correct as stated. For example, if S is the integral lattice in $\mathbf{R}^2$ and K is a square with sides parallel to the coordinate axes, then no homothetic copy of K can contain exactly 3 points of S. The difficulty is that the "equidistant sets" $C(a, b)$ used in the proof need not be nowhere dense, as asserted in the paper. The theorem and proof however can be salvaged by restricting K to be strictly convex. It also appears to be the case that the theorem is correct if "homothetic" is replaced by "similar" in the statement, without restricting K.

(Reçu le 5 décembre 1975)

G. D. Chakerian
Department of Mathematics
University of California
Davis, California 95616

vide-leer-empty