2. Fourier Séries Results

2. Fourier Series Results

Let the integrable function $f(x)$, $-\pi \leqslant x \leqslant \pi$, have period 2π, so that $f(x+2\pi) = f(x)$, and in particular $f(\pi) = f(-\pi)$, and assume that $0 < \alpha \leqslant 1$ and $p \geqslant 1$. Denote by Δf one of the three differences (it matters not which for our purposes)

$$f(x) - f(x-h), \quad f(x+h) - f(x), \quad f(x+h) - f(x-h).$$

If $\Delta f = O(|h|^{\alpha})$ we say either that $f(x)$ *belongs to* Lip α or that $f(x)$ *satisfies a Lipschitz condition with exponent* α. More generally, $f(x)$ is said to *belong to the Lipschitz class* Lip (α, p) if

$$\int_{-\pi}^{\pi} |\Delta f|^{p}\, dx = O(|h|^{\alpha p}).$$

In view of Hölder's inequality, a function of Lip (α, p) belongs to Lip (α, q) for all $1 \leqslant q < p$. Moreover, a function of Lip α clearly belongs to Lip (α, p) for all $p \geqslant 1$. In fact, the class Lip α may be viewed roughly as the limit of Lip (α, p) for $p = \infty$.

The classical complex Fourier series of $f(x)$ is defined by

$$f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{inx} \text{ where } c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, e^{-inx}\, dx.$$

Equivalently, if $c_n \equiv \frac{1}{2}(a_n - ib_n)$ for all n, then

$$f(x) \sim \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

with

$$\begin{matrix} a_n \\ b_n \end{matrix} = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \begin{matrix} \cos \\ \sin \end{matrix}\, nx\, dx.$$

For given integrable f, the series

$$\sum_{n=-\infty}^{\infty} |c_n|^{\gamma}$$

of moduli of the coefficients of these Fourier series may not converge for any finite $\gamma > 0$. If it does for certain γ, however, the convergence exponent

ρ of the Fourier coefficients is the infimum of these γ. For square-integrable f, we know that $\rho \leqslant 2$. (The above series, of course, need not be convergent for $\gamma = \rho$.)

The earliest result of interest to us here is the well-known theorem of Bernstein [2], [3], [4] (see also Bary [1], pp. 153-171, or Zygmund [37], pp. 240-243, for example) which we state as follows:

THEOREM 2.1. *If* $f(x)$ *is in* $\operatorname{Lip}\alpha$ *with* $\alpha > \frac{1}{2}$, *then* $\rho < 1$.

This result has a sharpened form due to Szász [26], namely:

THEOREM 2.2. *If* $f(x)$ *is in* $\operatorname{Lip}\alpha$, *then* $\rho = 1/(\alpha + 1/2)$,

and an even more general rendition due essentially to Szász [26] (the case $p = 2$), [27], Titchmarsh [28] (the corresponding theorem for transforms; see also [29]), and Hardy and Littlewood [19] (under the assumption $\alpha p > 1$):

THEOREM 2.3. *If* $f(x)$ *belongs to* $\operatorname{Lip}(\alpha, p)$, *then*

$$\rho = \begin{cases} \dfrac{1}{\alpha + 1 - 1/p} & 1 \leqslant p \leqslant 2 \\[2ex] \dfrac{1}{\alpha + 1/2} & p > 2. \end{cases}$$

For square-integrable f, this result only has relevance, of course, when $2\alpha p > 2 - p$.

We note in passing that since the class $\operatorname{Lip}(1, p)$, where $p > 1$, is equivalent to the collection of integrals of functions of the Lebesgue class L^p (Hardy and Littlewood [18], p. 599), Theorem 2.3 has as a special case the well-known result originally established by Tonelli [30]:

COROLLARY. *If* $f(x)$ *is absolutely continuous and its derivative* $f'(x)$ *belongs to* $L^p, p > 1$, *then* $\rho < 1$.

Other restrictions on $f(x)$, $-\pi \leqslant x \leqslant \pi$, are also of interest to us. Finite-valued functions are said to be of *bounded variation* if for all $N \geqslant 1$ and arbitrary choice of partition $-\pi \leqslant x_0 \leqslant x_1 \leqslant \dots \leqslant x_N \leqslant \pi$,

$$\sum_{n=1}^{N} | f(x_n) - f(x_{n-1}) | \leqslant B \text{ (const.)} < \infty .$$

Since $f(x)$ is in Lip $(1, 1)$ if and only if (iff) it is of bounded variation, no new results arise without at least some modest additional assumptions beyond mere bounded variation. One such set of combined restrictions leads to the following classical result first established by Zygmund [35] (see also Bary [1], Zygmund [37]):

THEOREM 2.4. *If* $f(x)$ *is of bounded variation and also in* Lip β *for some* $\beta > 0$, *then* $\rho < 1$.

Here also there is a sharpened form, this time due to Waraszkiewicz [31] (see also Zygmund [36]):

THEOREM 2.5. *If* $f(x)$ *is of bounded variation and also in* Lip β *for some* $\beta > 0$, *then* $\rho = 1/(1+\beta/2)$.

Other results, employing different sets of combined assumptions, can be established using the convexity property of the class Lip (α, p) (Hardy and Littlewood [20]), namely:

PROPERTY 1. If $f(x)$ belongs both to Lip (α, p) and to Lip (β, q), where $p < q$, then it belongs to Lip (γ, r) for all $p \leqslant r \leqslant q$, where

$$\gamma = \alpha \frac{p(q-r)}{r(q-p)} + \beta \frac{q(r-p)}{r(q-p)} .$$

In the limiting case $q = \infty$, where $f(x)$ is in Lip β, then

$$\gamma = \beta + (\alpha - \beta) \frac{p}{r} .$$

Interplaying this property with the earlier Theorem 2.3, we obtain the general

THEOREM 2.6. *If* $f(x)$ *belongs both to* Lip (α, p) *and to* Lip (β, q), *where* $p < q$, *then*

i) *for* $q \leqslant 2$,

$$\rho = \begin{cases} \dfrac{1}{\alpha + 1 - 1/p} & pq(\alpha - \beta) > q - p \\[2ex] \dfrac{1}{\beta + 1 - 1/q} & pq(\alpha - \beta) \leqslant q - p , \end{cases}$$

ii) *for* $p \leqslant 2 < q$,

$$\rho = \begin{cases} \dfrac{1}{\alpha + 1 - 1/p} & pq(\alpha - \beta) > q - p \\ \dfrac{2(q-p)}{q(2\beta + \alpha p + 1) - p(2\alpha + \beta q + 1)} & 0 < pq(\alpha - \beta) \leqslant q - p \\ \dfrac{1}{\beta + 1/2} & \alpha \leqslant \beta\,, \end{cases}$$

iii) *and for* $p > 2$,

$$\rho = \begin{cases} \dfrac{1}{\alpha + 1/2} & \alpha > \beta \\ \dfrac{1}{\beta + 1/2} & \alpha \leqslant \beta\,. \end{cases}$$

Theorem 2.5 is the special case of this result when $\alpha = p = 1, q = \infty$. Other special cases are:

Corollary 1. *If* $f(x)$ *is of bounded variation and also in* $\mathrm{Lip}(\beta, q)$ *for some* $\beta > 0, q \geqslant 1$, *then*

$$\rho = \begin{cases} 1 & \beta q < 1 \\ \dfrac{q}{\beta q + q - 1} & \beta q \geqslant 1, q \leqslant 2 \\ \dfrac{2(q-1)}{\beta q + 2q - 3} & \beta q \geqslant 1, q > 2\,; \end{cases}$$

Corollary 2. *If* $f(x)$ *belongs to* $\mathrm{Lip}(\alpha, p)$ *and also satisfies an ordinary Lipschitz condition with exponent* $\beta > 0$, *then*

$$\rho = \begin{cases} \dfrac{p}{\alpha p + p - 1} & p(\alpha - \beta) > 1, \quad p \leqslant 2 \\ \dfrac{2}{\beta(2-p) + \alpha p + 1} & 0 < p(\alpha - \beta) \leqslant 1, \quad p \leqslant 2 \\ \dfrac{1}{\alpha + 1/2} & \alpha > \beta, \quad p > 2 \\ \dfrac{1}{\beta + 1/2} & \alpha \leqslant \beta. \end{cases}$$

We note that $\rho < 1$ for $\beta q > 1$ in the first case, while for $p \leqslant 2$, $\alpha > \beta > (1-\alpha p)/(2-p)$ gives the same conclusion in the latter situation. Comparable results were observed by Hardy and Littlewood [20] and Waraszkiewicz [31].

Perhaps not surprisingly, the Corollary to Theorem 2.3 may be viewed as a special case of Corollary 2 above since when $\alpha p > 1$, functions in Lip (α, p) likewise belong to Lip $(\alpha - 1/p + 1/q, q)$ for all $q > p$ and hence are equivalent to functions in Lip $(\alpha - 1/p)$ (Hardy and Littlewood [19]). Alternatively, the earlier result can also be established using the following variant of one-half of the Hausdorff-Young Theorem (Hausdorff [21], Young [33], [34]; see also Hardy and Littlewood [17], Bary [1], Zygmund [37]) and the familiar relation between the Fourier coefficients of $f(x)$ and its derivatives $f^{(s)}(x)$, $s = 1, 2, \ldots$:

THEOREM 2.7. *If* $f(x)$ *is in* $L^p, p > 1$, *then*

$$\rho = \begin{cases} \dfrac{p}{p-1} & p \leqslant 2 \\ 2 & p > 2. \end{cases}$$

PROPERTY 2. If $f^{(s-1)}(x)$ is absolutely continuous for some positive integer s, then the Fourier coefficients c_{ns} of $f^{(s)}(x)$ are given by

$$c_{ns} = (in)^s c_n .$$

(Here, of course, we have made the tacit assumption that the periodic $f^{(r)}(x)$, $0 \leqslant r \leqslant s-1$, are all continuous in the wide-sense, i.e. for all x, so that in particular $f^{(r)}(\pi) = f^{(r)}(-\pi)$, $0 \leqslant r \leqslant s-1$.) Property 2 easily leads to

PROPERTY 3. If $f^{(s-1)}(x)$ is absolutely continuous for some positive integer s, and the convergence exponent of the Fourier coefficients of $f^{(s)}(x)$ is ρ_s, then

$$\rho = \frac{\rho_s}{1 + s\rho_s} .$$

Taken together, the above results finally yield the general

THEOREM 2.8. *If* $f^{(s-1)}(x)$ *is absolutely continuous for some positive integer* s, *and* $f^{(s)}(x)$ *belongs to* L^p, $p > 1$, *then*

$$\rho = \begin{cases} \dfrac{p}{p(s+1) - 1} & p \leqslant 2 \\ \dfrac{2}{1 + 2s} & p > 2. \end{cases}$$

In particular, for $s = 1$

$$\rho = \begin{cases} \dfrac{p}{2p - 1} & p \leqslant 2 \\ \dfrac{2}{3} & p > 2. \end{cases}$$

Any number of other deductions can be obtained by combining Theorem 2.8 with earlier results. We content ourselves with

THEOREM 2.9. *If* $f^{(s-1)}(x)$ *is absolutely continuous for some positive* s, *and if* $f^{(s)}(x)$ *is of bounded variation and also in* Lip (β, q) *for some* $\beta > 0$, $q \geqslant 1$, *then*

$$\rho = \begin{cases} \dfrac{1}{1 + s} & \beta q < 1 \\ \dfrac{q}{q(\beta + 1 + s) - 1} & \beta q \geqslant 1, \ q \leqslant 2 \\ \dfrac{2(q-1)}{q(\beta + 2 + 2s) - 3 - 2s} & \beta q \geqslant 1, \ q > 2; \end{cases}$$

THEOREM 2.10. *If* $f^{(s-1)}(x)$ *is absolutely continuous for some positive* s, *and if* $f^{(s)}(x)$ *belongs to* Lip (α, p) *and also satisfies an ordinary Lipschitz condition with exponent* $\beta > 0$, *then*

$$\rho = \begin{cases} \dfrac{p}{p(\alpha + 1 + s) - 1} & p(\alpha - \beta) > 1, p \leqslant 2 \\ \dfrac{2}{\beta(2-p) + \alpha p + 1 + 2s} & 0 < p(\alpha - \beta) \leqslant 1, p \leqslant 2 \\ \dfrac{1}{\alpha + s + 1/2} & \alpha > \beta, p > 2 \\ \dfrac{1}{\beta + s + 1/2} & \alpha \leqslant \beta. \end{cases}$$