§2. Kahane's set K_ξ.

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Let ξ be a real number, and let T_ξ be the (1/2)-trapezoid with bases

$$L = \{(x, 0): 0 \leqslant x \leqslant 1\} \quad \text{and} \quad L' = \{(x', 1): \xi \leqslant x' \leqslant \xi + 1/2\}.$$

Let T_{11}, T_{12}, T_{13}, and T_{14} be the four descendants of T_ξ labeled in such a way that the points

$$\alpha_{1k} = \min\{x: (x, 1/2) \in T_{1k}\}$$

satisfy $\alpha_{1k} < \alpha_{1(k+1)}$ for $k = 1, 2$, and 3. Then let T_{2k} ($k=1, 2, ..., 4^2$) be the 4^2 descendants of the trapezoids T_{1k} ($k=1, 2, 3, 4$) labeled in such a way that the points

$$\alpha_{2k} = \min\{x: (x, 1/2) \in T_{2k}\}$$

satisfy $\alpha_{2k} < \alpha_{2(k+1)}$ for $k = 1, 2, ..., 4^2-1$. Continuing inductively we arrive at the collection of (1/2)-trapezoids

$$T_{nk} \quad \text{for} \quad n = 1, 2, \dots \quad \text{and} \quad k = 1, 2, \dots, 4^n$$

where the T_{nk} ($k=1, 2, ..., 4^n$) are the descendants of the $T_{(n-1)k}$ ($k=1, 2, ..., 4^{n-1}$) and are labeled in such a way that the points

$$\alpha_{nk} = \min\{x: (x, 1/2) \in T_{nk}\}$$

satisfy $\alpha_{nk} < \alpha_{n(k+1)}$ for $k = 1, 2, ..., 4^n-1$.

If for each n we set

$$\alpha_{n(4^n+1)} = \max\{x: (x, 1/2) \in T_{n4^n}\},$$

then it follows from (A) in §1 that both

$$T_{nk}(1/2) = \{(x, 1/2): \alpha_{nk} \leqslant x \leqslant \alpha_{n(k+1)}\}$$

for each $k = 1, 2, ..., 4^n$, and for each n

$$T_\xi(1/2) = \bigcup_{k=1}^{4^n} T_{nk}(1/2).$$

Now, for each n, let $K_n = \bigcup_{k=1}^{4^n} T_{nk}$ and define the set

$$K_\xi = \bigcap_{n=1}^{\infty} K_n.$$

The following properties of the set K_ξ were established by Kahane [5]:

(I) K_ξ *is a compact set of 2-dimensional measure zero.*

(II) *Let any segment* τ *that extends from one base of* T_ξ *to the other be called "admissible". Then the following two properties hold:*

(i) *There are at most two admissible line segments in* K_ξ *that pass through any given point* $z \in T_\xi(1/2)$; *in fact, precisely one such segment passes through each* $z \in T_\xi(1/2)$ *that is not one of the* α_{nk}.

(ii) *For each* $\lambda \in [\xi-1, \xi+1/2] = \text{proj}\,[T_\xi]$, *there exists at least one point* $z \in T_\xi(1/2)$ *through which there passes an admissible line segment* τ *in* K_ξ *with* $\text{proj}\,[\tau] = \lambda$; *in fact, for all but countably many such* λ, *there exists only one such* z.

We now introduce a set-valued function Λ defined on the subsets A of $T_\xi(1/2)$ as follows:

$$\Lambda(A) = \{\text{proj}\,[\tau]: \tau \subset K_\xi \text{ is an admissible segment with } A \cap \tau \neq \phi\}.$$

Then K_ξ has the two additional properties:

(III) $\Lambda(A)$ *is nowhere dense whenever* A *is.*

(IV) $\Lambda(A)$ *is of linear (Lebesgue) measure zero whenever* A *is.*

Proof of (III). Let A be a nowhere dense subset of $T_\xi(1/2)$, and suppose $\Lambda(A)$ is dense on some subinterval I of the interval $\text{proj}\,[T_\xi]$. Then, according to (B) in §1, there exist integers n and k such that $\Lambda(A)$ is dense on $\text{proj}\,[T_{nk}]$. Since A is nowhere dense, there exist integers n' and k' such that $T_{n'k'}(1/2) \subset T_{nk}(1/2)$ and $A \cap T_{n'k'} = \phi$. By (B) of §1, it follows that

$$\Lambda(A) \cap \text{int}(\text{proj}\,[T_{n'k'}]) = \phi$$

(int $\equiv$ interior). This contradicts the fact that $\Lambda(A)$ is dense on $\text{proj}\,[T_{nk}]$, and property (III) is proved.

For the remainder of this paper, we use $\mu(A)$ and $\mu^*(A)$ to denote respectively the Lebesgue measure and the Lebesgue outer measure of the linear set A.

Proof of (IV). Let $A \subset T_\xi(1/2)$ and suppose $\mu(A) = 0$. If we set

$$\hat{A} = A - \{\alpha_{nk}: n = 1, 2, \ldots;\ k = 1, 2, \ldots, 4^n + 1\},$$

then in view of II (i) it suffices to show that the corresponding set $\Lambda(\hat{A})$ has linear measure zero.

Let $\mathscr{I}$ be the collection of all closed intervals on $T_\xi(1/2)$ of the form $[\alpha_{nk}, \alpha_{n(k+1)}]$. Then it is easy to verify that

$$\mu(\hat{A}) = \inf \sum_{j=1}^{\infty} \mu(I_j),$$

where the inf is taken over all sequences $\{I_j\}$ of intervals in $\mathscr{I}$ covering $\hat{A}$.

Let $\varepsilon > 0$. Then there is a sequence $\{I_j\}$ of intervals in $\mathscr{I}$ that cover $\hat{A}$ such that

$$\sum_{j=1}^{\infty} \mu(I_j) < \varepsilon.$$

For each index j, there exist integers n_j and k_j such that

$$I_j = T_{n_j k_j}(1/2);$$

hence, in view of (B) in §1, we have

$$\Lambda(\hat{A}) \subset \bigcup_{j=1}^{\infty} \operatorname{proj}[T_{n_j k_j}].$$

Furthermore, combining (A) and (B) of §1, we obtain

$$\mu(\operatorname{proj}[T_{n_j k_j}]) = 2\mu(I_j) \quad (j = 1, 2, \ldots).$$

Therefore,

$$\mu^*(\Lambda(\hat{A})) \leqslant \sum_{j=1}^{\infty} \mu(\operatorname{proj}[T_{n_j k_j}]) = 2 \sum_{j=1}^{\infty} \mu(I_j) < 2\varepsilon,$$

and property (IV) is proved.

§3. Proof of Theorem 1

For each integer n, let $\xi_n = 1 + 3n/2$ and set

$$\Delta_n = K_{\xi_n} \cap \{(x, y): \xi_n/2 \leqslant x \leqslant \xi_n/2 + 3/4 \quad \text{and} \quad 1/2 < y \leqslant 1\}.$$

Then set

$$\Delta_n^* = \{z - (1+i)/2 : z \in \Delta_n\} \quad (i = \sqrt{-1}),$$

and define the set

$$\Delta = \cup \{\Delta_n^* : n = 0, \pm 1, \pm 2, \ldots\}.$$

Let $\{z_n\}_{n=1}^{\infty}$ be a sequence of points in $H - \Delta$ whose derived set is R. Define the function f_0 on $\Delta \cup \{z_n\}_{n=1}^{\infty}$ by