Preface

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PREFACE

By means of an algebraic substitution, the so-called Tschirnhaus transformation, the general algebraic equation of the n-th degree $x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_{n-1} x + a_n = 0$ may be reduced to the form $y^n + b_4 y^{n-4} + b_5 y^{n-5} + \ldots + b_{n-1} y + 1 = 0$. Further attempts by algebraists to reduce the solution of the general algebraic equation to the solution of equations containing a smaller number of parameters remained unsuccessful for a long time (the problem of resolvents).

In his famous Mathematical Problems [1] Hilbert looked at this problem in a new way, formulating it as No. 13 in the following form: the impossibility of solving the general equation of the 7-th degree by means of functions of only two variables. To prove this Hilbert regarded it as possible to show that the equation of the 7-th degree $f^7 + xf^3 + yf^2 + zf + 1 = 0$ is not soluble by means of any continuous functions of only two variables.

Various mathematicians have understood the 13-th Problem differently and have attributed to it results of a different character.

Hilbert [3] found an algebraic substitution reducing the solution of the general algebraic equation of the 9-th degree to the solution of equations with 4 parameters. Hilbert proved also the existence of analytic functions of three variables not representable by superpositions of functions of only two variables. Ostrowski [2] constructed an analytic function of two variables not representable as a superposition of infinitely differentiable functions of one variable and arithmetic operations. The author [4] proved the existence of smooth functions of several variables not representable by superpositions of smooth functions of a smaller number of variables.

Bieberbach [5] attempted to prove that there exist continuous functions of three variables, not representable as a superposition of continuous functions of two variables. Not for nothing did Bieberbach call the 13-th Problem "unfortunate" (see [6]). Many years later, by the combined efforts of Kolmogorov [7], [9] and Arnol'd [8], the opposite was proved. So Hilbert's conjecture was shown to be false. By Kolmogorov's theorem any continuous function of several variables can be represented by means of a superposition of continuous functions of a single variable and the operation of addition.

Hilbert's 13-th problem gave rise to a great number of investigations in algebra and analysis, but the kernel of the problem never the less remains untouched. In this connection Lorentz [12] made an expressive analogy. The example of Peano of a mapping of an interval onto a square does not answer the question about the difference between an interval and a square. In the same way the theorem of Kolmogorov does not close the 13-th problem, but only makes it more interesting. It is known, for example, that superpositions of Kolmogorov's type, composed of smooth functions, do not even represent all analytic functions [48].

Thus, Hilbert's idea of proving the impossibility of solving the general equation of the 7-th degree by means of functions of only two variables can be developed in a more positive way. Results available at present do not contradict, for example, the possibility that the function $f(x, y, z)$ defined by the equation $f^7 + xf^3 + yf^2 + zf + 1 \equiv 0$ is not a finite superposition of analytic functions of two variables. On the other hand nobody has disproved that any algebraic function is a superposition of algebraic functions of a single variable and arithmetic operations.

This paper is a summary of the lectures given at the University of California in Los Angeles in April-May of 1977. Chapter I presents a survey of results, the remaining chapters are devoted to proofs.

Chapter 1. — Survey of results

The survey presented is based on the surveys [10]-[12], [33]-[35]. It also covers recent results:

Definition. We will say that a function $f = f(x_1, ..., x_n)$ is a superposition of the functions