§1. Superpositions of analytic functions

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$$\varphi^{(\alpha)}_{\beta_1, \beta_2, \ldots, \beta_\alpha}\left(U^{(\alpha+1)}_{\beta_1, \beta_2, \ldots, \beta_\alpha, 1}, U^{(\alpha+1)}_{\beta_1, \beta_2, \ldots, \beta_\alpha, 2}, \ldots, U^{(\alpha+1)}_{\beta_1, \beta_2, \ldots, \beta_\alpha, k}\right)$$
$$(\beta_i = 1, 2, \ldots, k, \ i = 1, \ldots, \alpha, \ \alpha = 0, 1, \ldots, s-1)$$

of k variables if f identically equals the function φ, defined by the equalities

$$\varphi = \varphi^{(0)}(U_1{}^{(1)}, U_2{}^{(1)}, \ldots, U_k{}^{(1)}),$$
$$U^{(\alpha)}_{\beta_1, \ldots, \beta_\alpha} = \varphi^{(\alpha)}_{\beta_1, \ldots, \beta_\alpha}\left(U^{(\alpha+1)}_{\beta_1, \beta_2, \ldots, \beta_\alpha, 1}, U^{(\alpha+1)}_{\beta_1, \beta_2, \ldots, \beta_\alpha, 2}, \ldots, U^{(\alpha+1)}_{\beta_1, \beta_2, \ldots, \beta_\alpha, k}\right)$$
$$\beta_i = 1, 2, \ldots, k, \ i = 1, 2, \ldots, \alpha, \ \alpha = 1, 2, \ldots, s-1,$$
$$U^{(s)}_{\beta_1, \beta_2, \ldots, \beta_s} = x_{j(\beta_1, \beta_2, \ldots, \beta_s)}.$$

The number s is called the order of superposition.

§ 1. *Superpositions of analytic functions*

In stating the 13-th Problem [1] Hilbert added that he had a rigorous proof of the fact that there exists an analytic function of three variables that cannot be obtained by a finite superposition of functions of only two arguments. Although he did not indicate exactly what kind of functions of two variables he had in mind, Hilbert was apparently thinking of analytic functions of two variables.

The existence of analytic functions of three variables not representable by means of superpositions of analytic functions of two variables is a simple fact and can be obtained from the following considerations. The partial derivatives of order k of a function represented by a superposition are defined by the derivatives of the functions composing the superposition. The number of different partial derivatives of order p of a function of two variables is equal to $\frac{p(p-1)}{1 \cdot 2}$. Consequently, the number of parameters defining the derivatives of order k of the superposition has order k^3 (s is fixed). On the other hand the number of different partial derivatives of order not greater than k for a function of three variables is of the order k^4. Hence for any s there exists a sufficiently large k such that one can find a polynomial of the k-th degree not representable by a superposition of order s of infinitely differentiable functions of two variables. The desired non-representable analytic function can be given as a sum of non-representable polynomials.

More general results in this direction were obtained by Ostrowski [2], who showed, in particular, that the analytic function of two arguments

$\xi(x, y) = \sum_{n=1}^{\infty} \frac{x^n}{n^y}$ is not a finite superposition of infinitely differentiable functions of one variable and algebraic functions of any number of variables.

The proof of this result is based on the fact that the function $\xi(x, y)$ does not satisfy any algebraic partial differential equation, that is, an equation of the form

$$\Phi\left(\xi, \frac{\partial \xi}{\partial x}, \frac{\partial \xi}{\partial y}, \ldots, \frac{\partial^{\mu+\lambda} \xi(x, y)}{\partial x^\mu \partial y^\lambda}\right) = 0, \quad \text{where} \quad \Phi$$

is a polynomial with constant coefficients in the function ξ and its partial derivatives up to a certain order. At the same, it is comparatively simple to prove that any function of two variables which is a finite superposition of infinitely differentiable functions of one variable and algebraic functions of any number of variables necessarily satisfies some algebraic partial differential equation. In the same paper, Ostrowski conjectured that the function $\xi(x, y)$ is not a superposition of continuous functions of one variable and algebraic functions of any number of variables (see the theorem of Kolmogorov [9]).

§ 2. *The problem of resolvents*

Algebraic equations up to the 4-th degree inclusive are soluble by radicals, that is, the roots of these equations can be represented as functions of the coefficients in the form of a superposition of arithmetic operations and functions of one variable of the form $\sqrt[n]{t}$ $(n = 2, 3)$. The general equation of the 5-th degree, is insoluble by radicals, as Abel and Galois showed. But since the general equation of the 5-th degree may be reduced by algebraic substitutions to the form $x^5 + tx + 1 = 0$, containing a single parameter t, we may say that a root of the general equation of the 5-th degree is also represented as a function of the coefficients in the form of superpositions of arithmetic operations and algebraic functions of one variable. The problem of resolvents can be formulated in terms of superpositions in the following way: to find, for any number n, the smallest number k such that a root of the general equation of the n-th degree as a function of the coefficients is represented in the form of a superposition of algebraic functions of k variables. In [3] Hilbert conjectured that for $n = 6, 7, 8$ the number k is 2, 3, 4, respectively. Hilbert's result [3] for an equation of the 9-th degree was all the more unexpected: a root of the general equation of the 9-th