§1. Certain improvements of Kolmogorov 's theorem

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LEMMA 2.3.3. *If* $\frac{n}{s} > \frac{n'}{s'}$ *then for any natural* k *the set* $\Omega_k \cap C_s(\mathscr{I}^n)$ *is nowhere dense in* $C_s(\mathscr{I}^n)$.

By lemma 2.3.1 and the theorem 2.2.1 for any natural k $H_\varepsilon(\Omega_k) \leqslant C\left(\frac{1}{\varepsilon}\right)^{n'/s'}$, where C does not depend on ε. Hence, it follows from the inequality $\frac{n}{s} > \frac{n'}{s'}$ and lemma 2.3.2 that the set $\Omega_k \cap C_s(\mathscr{I}^n)$ is nowhere dense in $C_s(\mathscr{I}^n)$.

Now to prove the theorem we have to notice only that the set of functions from $C_s(\mathscr{I}^n)$ representable by superpositions coincides with $\bigcup_{k=1}^{\infty}(\Omega_k \cap C_s(\mathscr{I}^n))$. By lemma 2.3.3 the sets $\{\Omega_k \cap C_s(\mathscr{I}^n)\}$ are nowhere dense and consequently the set of not representable functions is a set of second category.

CHAPTER 3. — SUPERPOSITIONS OF CONTINUOUS FUNCTIONS

In this chapter we present the proof of the theorem of Kolmogorov given by Kahane [36]. This proof which is based on Baire's theory contains a minimum of concrete constructions and shows that there exists a wide choice of inner functions for Kolmogorov's formula.

§ 1. *Certain improvements of Kolmogorov's theorem*

By the theorem of Kolmogorov any function defined and continuous on the cube $\mathscr{I}^n$ can be represented as

$$f(x_1, \dots, x_n) = \sum_{q=1}^{2n+1} g_q\Big(\sum_{p=1}^{n} \varphi_{p,q}(x_p)\Big),$$

where $\{\varphi_{p,q}\}$ are specially chosen continuous and monotonic functions which do not depend on f, and where $\{g_q\}$ are continuous functions.

Lorentz [12] has noticed that in the theorem of Kolmogorov the functions $\{g_q\}$ can be chosen independently of q. In fact, by adding constants to the functions $t_q = \sum_{p=1}^{n} \varphi_{p,q}(x_p)$ $(q = 1, \dots, 2n+1)$ one can make the ranges of the functions pairwise disjoint and consequently the functions $\{ t_q \}$ can be considered as the restrictions of a single function $\{ g_q \}$.

Sprecher [40] has shown that the functions $\{ \varphi_{p,q} \}$ can be chosen in the form $\varphi_{p,q}(x_p) = \lambda_p \varphi_q(x_p)$ where $\{ \lambda_p \}$ are constants and $\{ \varphi_q \}$-are continuous monotonic functions.

Thus any continuous function can be represented as

$$f(x_1, \ldots, x_n) = \sum_{q=1}^{2n+1} g\left(\sum_{p=1}^{n} \lambda_p \varphi_q(x_p)\right),$$

where the constants $\{ \lambda_p \}$ and the continuous monotone functions $\{ \varphi_q \}$ do not depend on f, and where g is a continuous function.

Kahane [36] has shown that such a representation is possible with almost every collection of constants $\{ \lambda_p \}$ and "quasi every" collection of continuous functions $\{ \varphi_q \}$. The precise statement of this theorem will be given below. Here we consider some further results concerning the formula of Kolmogorov.

Doss [38] has shown that for any continuous monotonic functions $\varphi_{p,q}$ $(p=1, 2;\ q=1, 2, 3, 4)$ there exists a continuous function $f(x_1, x_2)$ of two variables not representable as a superposition of the form $\sum_{q=1}^{4} g_q \left(\sum_{p=1}^{2} \varphi_{p,q}(x_p)\right)$, where $\{ g_q \}$ are continuous functions.

Bassalygo [39] succeeded in showing that for any continuous functions $\varphi_i(x_1, x_2)$ $(i=1, 2, 3)$ there exists a continuous function $f(x_1, x_2)$ that is not equal to any superposition of the form $\sum_{i=1}^{3} g_i(\varphi_i(x_1, x_2))$, where $\{ g_i \}$ are continuous functions.

Tihomirov showed that Kolmogorov's theorem can be generalized as follows: for any compact K of dimension n there exists a homeomorphic embedding $\Psi(x) = \{ \Psi_1(x), \ldots, \Psi_{2n+1}(x) \}$, $x \in K$ into $(2n+1)$-dimensional euclidean space such that any continuous function $f(x)$ on K can be represented in the form $f(x) = \sum_{i=1}^{2n+1} g_i(\Psi_i(x))$, where $\{ g_i \}$ are continuous functions of one variable.

In the same paper [36] Kahane has shown that there exist complex numbers λ_p $(p=1, \ldots, n)$ and complex valued functions φ_q $(q=1, \ldots, 2n+1)$ possessing the following properties.

1. The function φ_q is a monotonic continuous transformation of the real axis onto the circle $|t| = 1$ $(q=1, \ldots, 2n+1)$.

2. The function $t_q = \sum_{p=1}^{n} \lambda_p \varphi_q (x_p)$ maps the cube $\mathcal{J}^n$ into the circle $|t| = 1$.

3. The transformation Ψ given by the equalities $t_q = \sum_{p=1}^{n} \lambda_p \varphi_q (x_p)$ $(q = 1, ..., 2n+1)$ is one-to-one on $\mathcal{J}^n$.

4. For any function f continuous on $\mathcal{J}^n$ there exists a function $g(z)$ continuous on the disk $|z| \leqslant 1$, holomorphic inside that disk, and such that $f = \sum_{q=1}^{2n+1} g \left(\sum_{p=1}^{n} \lambda_p \varphi_q (x_p) \right)$.

The transformation Ψ gives an embedding of the cube $\mathcal{J}^n$ into the torus $|t| = 1$ $(q = 1, ..., 2n+1)$ such that any function continuous on the cube $\tilde{\mathcal{J}}^n = \Psi(\mathcal{J}^n)$ is represented in the form $f(t_1, ..., t_{2n+1}) = \sum_{q=1}^{2n+1} g(t_q)$, where g is a function holomorphic in the unit disk. This means in particular that any function continuous on $\tilde{\mathcal{J}}^n$ has an analytic extension to the polydisk $|t_q| \leqslant 1$ $(q = 1, ..., 2n+1)$.

§ 2. *The theorem of Kahane*

Let M be a complete metric space. We recall that a set is called a set of second category if it is the intersection of a countable family of open sets which are everywhere dense in M. By the theorem of Baire in a complete metric space no set of second category is empty. The massivity of such sets is characterized by the fact that the intersection of a countable family of sets of second category is again a set of second category and consequently is not empty.

We will say that a statement is true for quasi every element of M if it is true for a set of elements of second category.

Let us consider an example. Let Φ be the space with uniform norm consisting of all functions continuous and non-decreasing on the segment $\mathcal{J}^1$ $(0 \leqslant t \leqslant 1)$. It can be shown easily that quasi every element of Φ is a strictly increasing function.

In fact, any strictly increasing function belongs to any set defined as $\varphi(r') < \varphi(r'')$, where $r' < r''$ are fixed rational numbers. Any set defined by an inequality of that type is open and everywhere dense in Φ, and the set of all such sets is countable.