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$(q(x,y))\,p_k(x,y) \equiv 0$ on $e(q,t) \cap D$. If we had $C_k(t) = 0$ for some k, then it would turn out that $t \in \delta_k$. Consequently, $C_k(t) \neq 0$ for any k. We show that for every $t \in \delta^*$ the numbers $\{C_k(t)\}$ are uniquely determined. Assume the contrary. Then there are numbers $\{C_k'(t)\}$ $(\max |C_k'(t)| = 1)$ such that $\sum_{k=1}^{m} C_k'(q(x,y))\,p_k(x,y) = 0$ on $e(q,t) \cap D$ and $C_k \neq C_k'$ for some k. Then

$$\sum_{k \neq 1} [C_k(t)\,C_1'(t) - C_k'(t)\,C_1(t)]\,p_k(x,y) = \sum_{k \neq 1} C_k''(t)\,p_k(x,y) \equiv 0$$

on $e(q,t) \cap D$ and in addition, $C_k'' \neq 0$ for some k. Consequently, $t \in \delta_1$. So we have obtained a contradiction, and the uniqueness of the choice of the numbers $C_k(t)$ is proved. Further, we may regard $\{C_k(t)\}$ as single-valued functions of t on the portion δ^*. By Lemma 4.3.3, the functions $C_k(t)$ are continuous and, as noted above, $C_k(t) \neq 0$ for any $t \in \delta^*$. Then

$$p_1(x,y) = \sum_{k=2}^{m} -\frac{C_k(q(x,y))}{C_1(q(x,y))}\,p_k(x,y)\,, \quad (x,y) \in q^{-1}(\delta^*) \cap D\,.$$

Putting $f(t) = f_k(t) - \dfrac{C_k(t)}{C_1(t)} f_1(t)$, $t \in \delta^*$, we have $\sum_{k=2}^{m} f_k^*(q(x,y))\,p_k(x,y)$

$$= \sum_{k=1}^{m} f_k(q)\,p_k(x,y) - \sum_{k=2}^{m} \frac{C_k(q)}{C_1(q)}\,p_k(x,y)$$

$$= \sum_{k=2}^{m} f_k(q)\,p_k(x,y) + f_1(q)\,p_1(x,y)$$

$$= \sum_{k=1}^{m} f_k(q(x,y))\,p_k(x,y)\,, \quad (x,y) \in q^{-1}(\delta^*) \cap D\,.$$

This proves the lemma.

§ 4. *Reduction of linear superpositions to a form with independent terms*

We fix the continuous functions $p_i^k(x,y)$ and continuously differentiable functions $q_i(x,y)$ $(i = 0, 1, 2, \ldots, n;\ k = 1, 2, \ldots, m_i)$ $n \geqslant 2$, where $\{q_i(x,y)\}$ satisfy in D conditions (1) and (3) of Lemma 4.2.2, and we consider in D superpositions of the form

$$\sum_{i=0}^{n} \sum_{k=1}^{m_i} p_i^k(x,y)\,f_i^k(q_i(x,y))\,,$$

where $\{f_i^k(t)\}$ are arbitrary continuous functions of one variable.

We call a bounded closed region $G \subset D$ polyhedral if the boundary of G consists of a finite number of mutually non-intersecting simple closed contours that are unions of a finite number of segments of level curves of the functions $q_i(x, y)$ $(i = 1, 2, ..., n)$. Let $G \subset D$ be a polyhedral region. We denote by Γ_i the set of those $t \in q_i(G)$ for which the set $e(q_i, t) \cap G$ contains a segment of a level curve belonging to the boundary of G. For any i the set Γ_i consists of a finite number of points. By property (1) of the functions $\{q_i(x, y)\}$ for every i and for all points $t_0 \in q_i(G) \backslash \Gamma_i$ there exists $\lim\limits_{t \to t_0} e(q_i, t) \doteqdot e(q_0, t_0)$. If $t_0 \in \Gamma_i$, then the last assertion need not hold, but in any case there exists $\lim\limits_{t \to +t_0} e(q_i, t) \subset e(q_i, t_0)$ and $\lim\limits_{t \to -t_0} e(q_i, t) \subset e(q_i, t_0)$ where the limit is taken over the points $t \in q_i(G)$. Here the limit is understood in the sense of the distance $\rho(e(q_i, t), e(q_i, t_0))$.

LEMMA 4.4.1. *There is a region* $G \subset D$ *and a system of numbers* $\tau_i^k = 0$ *or* 1 $(i = 0, 1, 2, ..., n;\ k = 1, 2, ..., m_i)$ *such that*

(4) *for any* i *and for any continuous functions* $\{\varphi_i^k(t)\}$ *there exist continuous functions* $\{f_i^k(t)\}$ *such that in* G

$$\sum_{k=1}^{m_i} p_i^k(x, y)\, \varphi_i^k(q_i(x, y)) \equiv \sum_{k=1}^{m_i} \tau_i^k p_i^k(x, y) f_i^k(q_i(x, y));$$

(5*) *for any polyhedral region* $G^* \subset G$ *and any* i, *the set*

$$\{t : \lambda(t, G^*, q_i, p_i^{k_1}, ..., p_i^{k_s}) = 0\}$$

is nowhere dense in $q_i(G^*)$, *where*

$$k_1 = k_1(i), k_2 = k_2(i), ..., k_s = k_s(i)$$

is the set of all values of k *for which* $\tau_i^k = 1$.

Proof. If $i = 0$, then by (1) the set $q_0(D)$ consists of only one point. We choose a region $G_0 \subset D$ and number τ_0^k $(k = 1, 2, ..., m_0)$ such that in G_0 the functions $p_0^{k_1}, ..., p_0^{k_s}$ are a basis for the linear hull of the functions $\{p_0^k\}$ (condition (4) for $i = 0$) and in any region $G^* \subset G_0$ these functions are linearly independent (condition (5*) for $i = 0$). Let $G^* \subset D$ be an arbitrary polyhedral region. Then $\lambda(t, G^*, q, \{p_i^k\})$ as a function of t has, for any $i > 0$, a finite number of points of discontinuity (of the first kind) on the set $q_i(G^*)$, which consists of a finite number of segments (see Lemma 4.3.1). Hence it follows that if the set $\{t : \lambda(t, G^*, q_i, \{p_i^k\}) = 0\}$ is not nowhere dense on $q_i(G^*)$, then the function $\lambda(t) \equiv 0$ on some segment $\delta \subset q_i(G^*)$ not containing points of Γ_i. By Lemma 4.3.4, there is a segment $\delta^* \subset \delta$ such that in the expression $\sum_{k=1}^{m_i} p_i^k(x, y) f_i^k(q_i(x, y))$ one of the terms can be deleted, without narrowing the class of the functions representable in the region $q^{-1}(\delta^*) \cap G^*$ as superpositions of the given form. Carrying out all possible deletions we can find a region $G \subset G_0 \subset D$ for which the assertion of the lemma is satisfied.

A region $G \subset D$ is called regular if, firstly, it is polyhedral and, secondly, there is a number $\gamma_G > 0$ such that for every $i > 0$ and every $t \in q_i(G)$ the set $e(q_i, t) \cap G$ is the union of a finite number of simple arcs, each of which has length not less than γ_G. A point A of the boundary of the polyhedral region G is called a vertex if it belongs simultaneously to two segments of the level curves of $q_i(x, y)$ and $q_j(x, y)$ $(i \neq j)$ on the boundary of G. Every polyhedral region has a finite number of vertices.

LEMMA 4.4.2. *For every polyhedral region* G *and every neighbourhood* U *of the vertices of this region we can construct a regular region* $G^* \subset G$ *such that* $G \backslash U \subset G^*$.

Proof. Let $A_1, A_2, \ldots, A_r$ be the vertices of the polyhedral region G; $U_1, U_2, \ldots, U_r$ suitably small neighbourhoods of these vertices. Let $k_m = k_m(A_m)$ be the number of all those functions $\{ q_i(x, y) \}$ for each of which the level curve passing through the point A_m does not contain any other points of the set $U_m \cap G$. Let $q_{im}(x, y)$ be one of these functions. We put $k(G) \in q_i(G)$. If $k(G) = 0$, then for any i and any $t \in q_i(G)$ the length of any component of the set $e(q_i, t) \cap G$ is greater than zero and consequently the region G is regular. Suppose that $k(G) > 0$ and m such that $k_m \neq 0$.

We fix $\varepsilon > 0$ and put

$$G_{1m}^* = G \,|\, \{(x, y): |q_{i_m}(x, y) - q(A_m)| < \varepsilon\} \cap U_m .$$

If U_m and ε are sufficiently small, then inside U_m the region G_{1m}^* has two vertices A_m' and A_m'', while the region G has only one vertex A_m there, but $k_m(A_m') = k_m(A_m'') = k_m(A_m) - 1$. We now put $G_1^* = \cap\, G_{1m}^*$, where the intersection is taken over all m such that $k_m \neq 0$. Then $k(G_1^*) = k(G) - 1$. Repeating this construction $k(G)$ times, we obtain a polyhedral region G^* for which $G \backslash G^* \subset U$ and $k(G^*) = 0$. Consequently, G^* is regular. This proves the lemma.

LEMMA 4.4.3. *There exists a set* $G \subset D$, *a number* $\lambda > 0$, *and a set of numbers* $\tau_i^k = 0$ *or* 1 $(i = 0, 1, \ldots, n;\ k = 1, 2, \ldots, m_i)$ *such that condition* (4) *of Lemma* 4.4.1 *is satisfied, and also the conditions*

(5) *for every* i *and* $t \in q_i(G)$ *and for any functions* $\{f_i^k(t)\}$

$$\max_{(x,y)\in e(q_i,t)\cap G} \left| \sum_{k=1}^{m_i} \tau_i^k p_i^k(x, y) f_i^k(q_i(x, y)) \right| \geqslant \lambda \max_k \left| \tau_i^k f_i^k(t) \right|;$$

(6) G *is a regular region.*

Proof. By Lemma 4.4.1 there exists a region $G^* \subset D$ and a set of numbers τ_i^k such that for every polyhedral subregion $G^{**} \subset G^*$ and for every i the set $\{t : \lambda(t, G^{**}, q_i, p_i^{k_1}, \ldots, p_i^{k_s}) = 0\}$ is nowhere dense in $q_i(G^{**})$, where $k_1, k_2, \ldots, k_s$ is the set of all values of k for which $\tau_i^k = 1$; moreover, on the set G^*, for any i the property (4) of Lemma 4.4.1 is satisfied. In order not to change the notation unnecessarily, we assume that all $\tau_i^k = 1$. We now construct a system of regular regions $G_0 \supset G_1 \supset G_2 \supset \ldots \supset G_n = G$, having the following property: for every $j \leqslant i$, $\inf_{t\in q_j(G_i)} \lambda(t, G_i, q_j, \{p_j^k\}) \geqslant \lambda_i > 0$. For G_0 we choose any regular region $G_0 \in G^*$. Suppose that the regular regions $G_0, G_1, \ldots, G_{i-1}$ have been constructed. We now construct the set G_i. We denote by α_δ the set $\{t : \lambda(t, q_i, G_{i-1}, \{p_i^k\}) > \delta\}$. Since the functions $\lambda(t, q_i, G_{i-1}, \{p_i^k\})$, have only finitely many points of discontinuity (of the first kind) on the set $q_i(G_{i-1})$, which consists of a finite number of segments (see Lemma 4.3.1), any component of α_δ is either an interval, or a half-interval, or a segment, or a point. Suppose that the set $\alpha_\delta^N \subset \alpha_\delta$ consists of the N longest components of non-zero length of the set α_δ (if α_δ has only N_0 $(< N)$ components of non-zero length, then let $\alpha_\delta^N = \alpha_\delta^{N_0}$). We denote by $\bar{\alpha}_\delta^N$ the closure of the set α_δ^N. We put $G_{i-1}^* = G_{i-1} \cap q_i^{-1}(\bar{\alpha}_\delta^N)$. We fix $\varepsilon > 0$. Since G_{i-1} is regular, for every j the length of any component of $e(q_j, t) \cap G_{i-1}$ is greater than $\gamma_G > 0$. And since the set $\{t : \lambda(t, q, G_{i-1}, \{p_i^k\}) = 0\}$ is nowhere dense in $q_i(G_{i-1})$, for sufficiently small δ and sufficiently large N the set G_{i-1}^* forms a $\varepsilon/2$-net on every set $e(q_j, t) \cap G_{i-1}$, $j < i$. The set G_{i-1}^* is a polyhedral region. We denote by $U(\varepsilon)$ the set of points (x, y) each of which is at a distance of no more than $\varepsilon/4$ from one of the vertices of the set G_{i-1}^*. By Lemma 4.4.2 there exists a regular region $G_i \subset G_{i-1}^*$ such that $G_{i-1}^* \backslash G_i \subset U(\varepsilon)$. The set G_i forms an ε-net on every set $e(q_j, t) \cap G_{i-1}, j < i$ and forms an $\varepsilon/2$-net on every set $e(q_i, t) \cap G_{i-1}^*$. By Lemma 4.3.2, for sufficiently small ε,

$$\lambda_i = \min_{j \leq i} \inf_{t \in q_j(G_i)} \lambda(t, G_i, q_j, \{p_i^k\}) > \frac{1}{2} \min\left\{\frac{\delta}{2}, \min_{j<i} \lambda_j\right\}.$$

Thus, the regular regions $G_1, G_2, \ldots, G_n$ can be constructed. The regular region $G = G_n$ satisfies all the requirements of our lemma ($\lambda = \lambda_n$), which is now proved.

§ 5. *The set of linear superpositions in the space of continuous functions is closed*

THEOREM 4.5.1. *Suppose that continuous functions* $p_m(x, y)$ *and continuously differentiable functions* $q_m(x, y)$ $(m = 1, 2, \ldots, N)$ *are fixed. Then in any region* D *of the plane of the variables* x, y. *there exists a closed subregion* $G \subset D$ *such that the set of superpositions of the form*

$$\sum_{m=1}^{N} p_m(x, y) f_m(q_m(x, y)),$$

where $\{f_m(t)\}$ *are arbitrary continuous functions, is closed (in the uniform metric) in the set of all functions continuous on the set* G.

By Lemma 4.2.2 and 4.4.3 we can find a subset $G \subset D$, determine constants $\gamma > 0$ and $\lambda > 0$, and renumber the functions $\{p_m(x, y)\}$ and $\{q_m(x, y)\}$ with two indices so that the functions obtained after the renumbering, $\{p_i^k(x, y)\}$ and $\{q_i^k(x, y)\}$ $(i = 0, 1, 2, \ldots, n;\ k = 1, 2, \ldots, m_i;\ \sum_{i=0}^{n} m_i \leq N)$ that is, some functions may be omitted in the renumbering) satisfy conditions (1), (2), (3) of Lemma 4.2.2, and also the conditions:

(4′) for any continuous functions $\{f_m(t)\}$ there exists continuous functions $\{f_i^k(t)\}$ such that on G

$$\sum_{m=1}^{N} p_m(x, y) f_m(q_m(x, y)) = \sum_{i=0}^{n} \sum_{k=1}^{m_i} p_i^k(x, y) f_i^k(q_i^k(x, y));$$

(5′) for every i and $t \in q_i^1(G)$ and for any functions $\{f_i^k(t)\}$

$$\max_{(x,y) \in e(q_i^1, t) \cap G} \left| \sum_{k=1}^{m_i} p_i^k(x, y) f_i^k(q_i^1(x, y)) \right| \leq \lambda \max_k |f_i^k(t)|;$$

(6′) G is a regular region with respect to the functions $\{q_i^k(x, y)\}$.