§6. The set of linear superpositions in the space of continuous functions is nowhere dense

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is satisfied, where the constant C does not depend on the functions $\{f_i^k(t)\}$. Since F is a bounded set, there exists a subsequence of suffixes $n_1, n_2, \ldots$ such that for any $i = 0, 1, \ldots, n$; $k = 1, 2, \ldots, m_i$; $j = 1, 2, \ldots, s_i$ the numerical sequence $f_{i,n_\nu}^k \to C_{k,i,j}$ as $\nu \to \infty$. From this and the previous inequality it follows that $\{f_{i,n_\nu}^k(t)\} \in F\,(\nu = 1, 2, \ldots)$ is a Cauchy sequence, because it is known that the sequence $f_n(x, y) \in T(F)$ is Cauchy sequence. Consequently there exists an element $\{f_i^k(t)\} \in B$ such that $\|\{f_i^k(t) - f_{i,n_\nu}^k(t)\}\|_B \to 0$. Since F is a closed set, $\{f_i^k(t)\} \in F$. The operator $T\colon B \to C(G)$ is bounded. Therefore $T(\{f_i^k(t)\}) = f(x, y)$. Consequently $f(x, y) \in T(F)$. This proves the lemma.

The following lemma from the theory of linear operators [28] turns out to be useful.

LEMMA 4.5.3. *Let B_1 and B_2 be Banach spaces. If a linear operator $T\colon B_1 \to B_2$ maps bounded closed sets of B_1 onto closed sets of B_2, then its domain of values is closed.*

Proof of Theorem 4.5.1. The set of superpositions of the form $\sum_{m=1}^{N} p_m(x, y) f_m(g_m(x, y))$ coincides on G with the set of superpositions of the form $\sum_{i=0}^{n} \sum_{k=1}^{m_i} p_i^k(x, y) f_i^k(q_i^1(x, y))$. By Lemma 4.5.2 and 4.5.3 the set of the latter superpositions is closed in the space $C(G)$. This proves the theorem.

§ 6. *The set of linear superpositions in the space of continuous functions is nowhere dense*

THEOREM 4.6.1. *For any continuous functions $p_m(x, y)$ and continuously differentiable functions $q_m(x, y)$ $(m = 1, 2, \ldots, N)$ and any region D of the plane of the variables x, y the set of superpositions of the form*

$$\sum_{m=1}^{N} p_m(x, y) f_m(q_m(x, y)),$$

where $\{f_m(t)\}$ are arbitrary continuous functions, is nowhere dense in the space of all functions continuous in D with uniform convergence.

By Lemma 4.2.2 we can find a subregion $G^* \subset D$, determine a constant $\gamma^* > 0$, and renumber the functions $\{q_m(x, y)\}$, with two indices so that the functions $\tilde{q}_i^k(x, y)$ $(i=0, 1, 2, ..., \tilde{n}; k=1, 2, ..., \tilde{m}_i; \sum_{i=0}^{\tilde{n}} \tilde{m}_i=N)$ obtained after the renumbering satisfy conditions (1), (2), (3) of Lemma 4.2.2. We now fix the point $(x_0, y_0) \in G^*$ and the number v so that the line $(y-y_0) + v(x-x_0) = 0$ does not touch at any of the level curves of the functions $\tilde{q}_i^k(x, y)$ $(i=1, 2, ..., \tilde{n})$ that pass through (x_0, y_0). Let $G^{**} \subset G^*$ be a disc with centre at (x_0, y_0) and radius small enough so that the $\{\tilde{q}_i^k(x, y)\}$ and $q_{N+1}(x, y) = y + vx$ satisfy condition (3) of Lemma 4.2.2 with some constant $\gamma^{**} > 0$. We put $p_{N+1}(x, y) = 1$. By Lemma 4.4.3 we can find a set $G \subset G^{**}$, determine a constant $\lambda > 0$, and again renumber the functions $p_m(x, y)$ and $q_m(x, y)$ $(m=1, 2, ..., N+1)$ with two indices so that the functions $p_i^k(x, y)$ and

$$q_i^k(x, y) \quad (i=0, 1, 2, ..., n+1\,; \; k=1, 2, ..., m_i\,; \; \sum_{i=0}^{n+1} m_i \leqslant N+1)$$

that is, some functions may be omitted in the renumbering) obtained after the renumbering satisfy conditions (1)-(3) of Lemma 4.2.2, conditions (4′)-(6′) of § 5, and the condition

$$7 \quad m_{n+1} = 1\,, \; p_{N+1}^1 = p_{N+1}(x, y) = 1\,, \; q_{N+1}^1 = q_{N+1}(x, y) = y + vx\,.$$

Let L be the linear space consisting of all system of functions $\{f_i^k(t)\}$ defined and continuous on the sets $\{q_i^1(G)\}$ and satisfying the condition

$$\sum_{i=0}^{n+1} \sum_{k=1}^{m_i} p_i^k(x, y) f_i^k(q_i^1(x, y)) \equiv 0 \quad \text{in} \quad G\,.$$

LEMMA 4.6.1. *L is a finite-dimensional linear space.*

Proof. By Lemma 4.5.1, in the sets $\{q_i^1(G)\}$ we can select a subset consisting of a finite number of points $\{t_{i,j}\}$ such that, if $\{f_i^k(t)\} \in L$ and $f_i^k(t_{i,j}) = 0$ for all k, i, j then $f_i^k(t) \equiv 0$ on $q_i^1(G)$ for all i, k. Thus, the set of functions $\{f_i^k(t)\}$ is completely determined by a finite set of parameters $\{f_i^k(t_{i,j})\}$. Consequently the dimension of the space L is finite. This proves the lemma.

LEMMA 4.6.2. *There exists a natural number μ such that in D the polynomial $(y+vx)^\mu = Q(x, y)$ is not equal to any superposition of the form $\sum_{m=1}^{N} p_m(x, y) f_m(q_m(x, y))$, where $\{f_m(t)\}$ are arbitrary continuous functions.*

Proof. We denote by Φ the space of functions of the form $f(y+vx) = f^{1}_{n+1}(q^{1}_{n+1}(x, y))$ that are representable on G by superpositions of the form $[\sum_{m=1}^{N} p_m(x, y) f_m(q_m(x, y))]$. Or, what comes to the same thing (see properties (4′) and (7)), of the form $[\sum_{i=0}^{n} \sum_{k=1}^{m_i} p_i^k(x, y) f_i^k(q_i^k(x, y))]$. Thus, functions of Φ satisfy the relation $\sum_{i=0}^{n+1} \sum_{k=1}^{m_i} p_i^k(x, y) f_i^k(q_i^k(x, y)) \equiv 0$ in G. Consequently the linear space Φ is naturally embedded in L. Since L is finite-dimensional (Lemma 4.6.1), Φ is also finite-dimensional. Let l be the dimension of Φ. Since the polynomials $(y+vx), (y+vx)^2, \ldots, (y+vx)^{l+1}$ are linearly independent, at least one of them $Q(x, y) = (y+vx)^{\mu}$ is not equal to any superposition of the form under discussion on G or, consequently, in D. This proves the lemma.

Proof of Theorem 4.6.1. By Lemma 4.6.2 the set of superpositions of the form given in Theorem 4.6.1 does not exhaust all continuous functions on G. Consequently, by Theorem 4.5.1, the set of these superpositions is a closed linear subspace of $C(G)$. Hence we conclude that the set of superpositions under discussion is nowhere dense in $C(G)$, nor consequently in $C(D)$. This proves the theorem.

COROLLARY 4.6.1. *For any continuous functions* $p_m(x_1, x_2, \ldots, x_n)$ *and continuously differentiable functions* $q_m(x_1, x_2, \ldots, x_n)$ $(m = 1, 2, \ldots, N)$ *and any region* D *of the space of the variables* $(x_1, x_2, \ldots, x_n)$ *the set of superpositions of the form*

$$\sum_{m=1}^{N} p_m(x_1, x_2, \ldots, x_n) f_m(q_m(x_1, x_2, \ldots, x_n), x_2, x_3, , x_{n-1}),$$

where $\{f_m(t, x_2, x_3, \ldots, x_{n-1})\}$ *are arbitrary continuous functions of* $(n-1)$ *variables, is nowhere dense in the space of all functions continuous in* D *with uniform convergence.*