3. Formulation of Siegel's Theorem

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of a compact set K. The right-hand side of the above formula reduces to the volume of K, while the left-hand side gives the mean value of

$$\text{card}\,(L - \{0\} \cap K)\,,$$

as L varies over all $\mathbf{Z}$-lattices in $\mathbf{R}^n$ with volume 1.

We turn now to the adelic mean value formula. Let G be a linear algebraic group defined over $\mathbf{Q}$, and let X be an algebraic homogeneous space for G, defined over $\mathbf{Q}$. For $\xi \in X$, let $G_\xi = \{g \in G : g\xi = \xi\}$. We assume that

a) X has at least one rational point

b) for any $\xi \in X_{\mathbf{C}}$, both $G_{\mathbf{C}}$ and $(G_\xi)_{\mathbf{C}}$ have finite fundamental groups

c) for any extension field K of $\mathbf{Q}$, G_K acts transitively on X_K.

We then have the following result.

THEOREM (Ono [2]). There are canonical measures on the adele spaces G_A and X_A such that, given any continuous function Φ on X_A with compact support,

$$(A) \qquad \frac{\displaystyle\int_{G_A/G_{\mathbf{Q}}} \sum_{x \in X_{\mathbf{Q}}} \Phi(gx)\, dg}{\tau(G_\xi)} = \int_{X_A} \Phi(x)\, dx\,,$$

where ξ is any element of $X_{\mathbf{Q}}$, and $\tau(G_\xi)$ = the invariant measure of $(G_\xi)_A / (G_\xi)_{\mathbf{Q}}$. The analogy to the previous mean value theorem is clear in the cases when $\tau(G) = \tau(G_\xi)$.

3. FORMULATION OF SIEGEL'S THEOREM

Let S and T be square matrices with integral entries of size m and n, respectively. We assume that both are positive definite. For any matrix x, denote $S[x] = {}^t xSx$ (when defined). Let $A(S, T)$ = the number of integral $m \times n$ matrices x such that $S[x] = T$. For each positive integer q, let $A_q(S, T)$ = the number of integral $m \times n$ matrices x, mod q, such that $S[x] \equiv T \pmod{q}$.

A positive definite integral matrix S' is said to be in the same class as S if $S' = S[U]$, for some $U \varepsilon SL(m, \mathbf{Z})$. S' is in the same genus as S if for each q, there exists $U \varepsilon SL(m, \mathbf{Z})$ such that $S' \equiv S[U] \pmod{q}$. Let $S_1, \ldots, S_h$ be the representatives of the classes in genus (S). Let $E(S_i)$ = the finite group consisting of all $U \varepsilon SL(m, \mathbf{Z})$ such that $S_i[U] = S_i$, and put

$e_i = 1 / \# E(S_i)$, where $\#$ denotes cardinality. We now define the "number of representations of T by the genus of S" as

$$A(\text{genus}(S), T) = \frac{e_1 A(S_1, T) + \ldots + e_h A(S_h, T)}{e_1 + \ldots + e_h}.$$

Now S is a real symmetric matrix, and so we may view it as a point in $\mathbf{R}^{n_1}$, where $n_1 = n(n+1)/2$. Similarly, T is a point in $\mathbf{R}^{m_1}$. Let dt be the usual measure in $\mathbf{R}^{m_1}$, and let dx be the usual measure in the real vector space of $m \times n$ matrices. Given $\varepsilon > 0$, let B_ε denote the ε-neighborhood of T in $\mathbf{R}^{m_1}$, and let C_ε denote the set of $x \varepsilon M_{m \times n}(\mathbf{R})$ satisfying $S[x] \varepsilon B_\varepsilon$. Then B_ε and C_ε are open sets with compact closure, and the following limit is known to exist:

$$A_\infty(S, T) = \lim_{\varepsilon \to 0} \int_{C_\varepsilon} dx \Big/ \int_{B_\varepsilon} dt.$$

THEOREM (Siegel [4]). For $m - n \geqslant 3$,

$$(S) \qquad A(\text{genus}(S), T) = A_\infty(S, T) \lim_q \frac{A_q(S, T)}{q^{mn - (n+1)n/2}}.$$

4. Derivation of Siegel's Theorem

Let $G = \{g \in SL(m) : S[g] = S\}$, and let $X = \{x \in M_{m \times n} : S[x] = T\}$. If $m \geqslant 4$, both $G_{\mathbf{C}}$ and $G_{\xi \mathbf{C}}$ have fundamental groups of order 2. Condition (c) of § 2 is the classical Witt theorem for (G, X). We assume that $X_{\mathbf{Q}}$ is nonempty.

We will show that (A) implies (S). This reduces Siegel's theorem to the computation of the Tamagawa number $\tau(G)$.

Let Φ_∞ = the constant function 1 on $X_{\mathbf{R}}$, and let Φ_p = the characteristic function of $X_{\mathbf{Z}_p}$ in $X_{\mathbf{Q}_p}$. Then $\Phi = \Phi_\infty \cdot \prod \Phi_p$ is the characteristic function of $X_{S_\infty} = X_{\mathbf{R}} \cdot \prod X_{\mathbf{Z}_p}$ in X_A. Because of the positive definiteness of S, Φ has compact support.

Consider the right-hand side of formula (S). Siegel has shown that there exists an algebraic gauge form dx on X such that $A_\infty(S, T) = \int_{X_{\mathbf{R}}} dx_\infty$, and

$$\lim_q \frac{A_q(S, T)}{q^{mn - (n+1)n/2}} = \prod_p \int_{X_{\mathbf{Z}_p}} dx_p,$$

where dx and dx_p are the positive measures induced on $X_{\mathbf{R}}$ and $X_{\mathbf{Q}_p}$ by dx.