

## 2. Proof of theorem B

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **23 (1977)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **25.05.2024**

### Nutzungsbedingungen

Die ETH-Bibliothek ist Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Inhalten der Zeitschriften. Die Rechte liegen in der Regel bei den Herausgebern.

Die auf der Plattform e-periodica veröffentlichten Dokumente stehen für nicht-kommerzielle Zwecke in Lehre und Forschung sowie für die private Nutzung frei zur Verfügung. Einzelne Dateien oder Ausdrucke aus diesem Angebot können zusammen mit diesen Nutzungsbedingungen und den korrekten Herkunftsbezeichnungen weitergegeben werden.

Das Veröffentlichen von Bildern in Print- und Online-Publikationen ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Die systematische Speicherung von Teilen des elektronischen Angebots auf anderen Servern bedarf ebenfalls des schriftlichen Einverständnisses der Rechteinhaber.

### Haftungsausschluss

Alle Angaben erfolgen ohne Gewähr für Vollständigkeit oder Richtigkeit. Es wird keine Haftung übernommen für Schäden durch die Verwendung von Informationen aus diesem Online-Angebot oder durch das Fehlen von Informationen. Dies gilt auch für Inhalte Dritter, die über dieses Angebot zugänglich sind.

**THEOREM B.** Let  $m, \chi, f, \psi$  be as above. We put

$$q = \prod_{\substack{p|m \\ p \nmid f}} p, \quad R = \frac{m}{fq}$$

where  $p$  denotes rational primes. Let the multiplicative function  $g$  be defined by

$$g(n) = \mu((n, q)) \varphi((n, q)) \bar{\psi}(n).$$

Then we have:

$$\left. \begin{aligned} \mathcal{G}(\alpha, \chi) &= 0 \text{ if } R \nmid \alpha \\ \mathcal{G}(Rn, \chi) &= \mu(q) \psi(q) \tau(\psi) R g(n), \quad n = 1, 2, \dots \end{aligned} \right\} \quad (3)$$

## 2. PROOF OF THEOREM B

For a Dirichlet character  $\chi \pmod{m}$  let the function  $L(s, \chi)$  be given by

$$L(s, \chi) = \sum_1^{\infty} \chi(n) n^{-s}, \quad \operatorname{Re} s > 1.$$

The series defines an analytic function for  $\operatorname{Re} s > 1$ , which can be extended to a meromorphic function on the whole complex plane, with at most one simple pole at  $s = 1$ . If  $\chi$  is primitive, then  $L(s, \chi)$  satisfies the equation

$$L(1-s, \chi) = m^{s-1} (2\pi)^{-s} \Gamma(s) \left( e^{-\frac{\pi i s}{2}} + \chi(-1) e^{\frac{\pi i s}{2}} \right) \tau(\chi) L(s, \bar{\chi}). \quad (4)$$

Because of (1), this can also be written, for  $\operatorname{Re} s > 1$ , as

$$L(1-s, \chi) = m^{s-1} (2\pi)^{-s} \Gamma(s) \left( e^{-\frac{\pi i s}{2}} + \chi(-1) e^{\frac{\pi i s}{2}} \right) \sum_1^{\infty} \mathcal{G}(n, \chi) n^{-s}. \quad (5)$$

Whereas (4) holds only for primitive characters, (5) turns out to be valid in the general case. In fact, a much more general formula is proved in [3], th. 6.1; if we put there  $x = \alpha = 0$ ,  $\alpha_n = \chi(n)$  and observe that  $\mathcal{G}(-n, \chi) = \chi(-1) \mathcal{G}(n, \chi)$ , we get (5) immediately. But also most of the classical (= non-adelic) proofs of (4) will give (5) after very small changes. The only use of the primitivity of  $\chi$  in these proofs is that they replace  $\mathcal{G}(n, \chi)$  by  $\bar{\chi}(n) \tau(\chi)$  at some stage (See for instance [7]).

Now let  $\chi, m, \psi, f$  be as in the theorem. We have, by the Euler-product,

$$\begin{aligned} L(1-s, \chi) &= L(1-s, \psi) \prod_{p|q} \left( 1 - \frac{\psi(p)}{p^{1-s}} \right) \\ &= L(1-s, \psi) \mu(q) q^{s-1} \psi(q) \prod_{p|q} (1 - p \bar{\psi}(p) p^{-s}) \end{aligned} \quad (6)$$

Formula (4) is valid if we write  $f$  for  $m$  and  $\psi$  for  $\chi$ . Eliminating  $L(1-s, \chi)$  and  $L(1-s, \psi)$  out of the equations (4), (5), (6), and taking into account that  $\chi(-1) = \psi(-1)$ , we obtain after an easy calculation, for  $\operatorname{Re} s > 1$ :

$$\sum_1^\infty \mathcal{G}(n, \chi) n^{-s} = R \tau(\psi) \mu(q) \psi(q) R^{-s} L(s, \bar{\psi}) \prod_{p|q} \left(1 - \frac{p\bar{\psi}(p)}{p^s}\right) \quad (7)$$

Using Euler's product, we have for  $\operatorname{Re} s > 1$

$$\begin{aligned} L(s, \bar{\psi}) \prod_{p|q} (1 - p\bar{\psi}(p) p^{-s}) \\ = \prod_{p \nmid q} \sum_{k=0}^\infty \bar{\psi}(p^k) p^{-ks} \cdot \prod_{p|q} (1 + (1-p) \sum_{k=1}^\infty \bar{\psi}(p^k) p^{-ks}) \\ = \prod_p \sum_{k=0}^\infty \bar{\psi}(p^k) \mu((p^k, q)) \varphi((p^k, q)) p^{-ks} \\ = \prod_p \sum_{k=0}^\infty g(p^k) p^{-ks} = \sum_1^\infty g(n) n^{-s}. \end{aligned}$$

If we put that into (7) we get

$$\sum_1^\infty \mathcal{G}(n, \chi) n^{-s} = R \tau(\psi) \mu(q) \psi(q) \sum_1^\infty g(n) (Rn)^{-s},$$

and we obtain (3) by comparing the coefficients.

3. In this section we show that one can prove theorem  $A$  using theorem  $B$  and conversely. Let  $K(\alpha)$  resp.  $H(\alpha)$  be the right hand side of (2) resp. (3). We want to prove  $K(\alpha) = H(\alpha)$ . First we show that  $K(\alpha) \neq 0$  iff  $H(\alpha) \neq 0$ .

Now  $H(\alpha) \neq 0$  iff  $R|\alpha$  and  $(\alpha R^{-1}, f) = 1$ , and this is equivalent to

$$(\alpha f q, f m) = m \quad (8)$$

On the other hand  $K(\alpha) \neq 0$  iff the four conditions hold

$$(\alpha, m)f | m \quad (9)$$

$$\frac{m}{(\alpha, m)f} \text{ is squarefree} \quad (10)$$

$$\left( \frac{m}{(\alpha, m)f}, f \right) = 1 \quad (11)$$

$$\left( \frac{\alpha}{(\alpha, m)}, f \right) = 1 \quad (12)$$

At several places in the following we will use that  $q$  is squarefree,  $(f, q) = 1$  and the prime divisors of  $m$  are precisely the prime divisors of  $fq$ .

Let us assume (8). Then  $f(\alpha, m) \mid f(\alpha q, m) = m$ . Also  $\frac{m}{f(\alpha, m)} = \frac{(\alpha q, m)}{(\alpha, m)} \mid q$ . This proves (9) and (10). From  $f = m/(\alpha q, m)$  we have  $(\alpha/(\alpha, m), f) = (\alpha/(\alpha, m), m/(\alpha q, m)) \mid (\alpha/(\alpha, m), m/(\alpha, m)) = 1$ .

This proves (12). Finally

$$\left( \frac{m}{(\alpha, m)f}, f \right) = \left( \frac{f(\alpha q, m)}{(\alpha, m)f}, f \right) \mid (q, f) = 1,$$

proving (11).

Conversely assume (9)-(12). From (10) and (11) we infer that  $\frac{m}{f(\alpha, m)} \mid q$ .

This implies

$$m \mid m \left( \frac{\alpha}{(\alpha, m)}, f \right) = \left( \frac{m\alpha f}{f(\alpha, m)}, mf \right) \mid (q\alpha f, mf). \quad (13)$$

Also

$$f(m, q\alpha) = f(\alpha, m) \left( \frac{m}{(\alpha, m)}, q \frac{\alpha}{(\alpha, m)} \right) = f(\alpha, m) \left( \frac{m}{(\alpha, m)}, q \right).$$

In the last term, the numbers  $f$  and  $(m/(\alpha, m), q)$  both divide  $m/(\alpha, m)$ , because of (9), and they are coprime, hence their product divides  $m/(\alpha, m)$ . This gives  $f(m, q\alpha) \mid m$ . Together with (13) this implies (8).

It remains to prove that  $H(\alpha) = K(\alpha)$  for  $\alpha = Rn$ ,  $(n, f) = 1$ . We have  $m_0 = m/(\alpha, m) = fq/(n, fq) = fq/(n, q)$ . Hence

$$\begin{aligned} R &= \frac{m}{fq} = \frac{\varphi(m)}{\varphi(fq)} = \frac{\varphi(m)}{\varphi(f)\varphi(q)} = \frac{\varphi(m)}{\varphi(f)\varphi((n, q))\varphi(\frac{q}{(n, q)})} \\ &= \frac{\varphi(m)}{\varphi((n, q))\varphi(m_0)} \end{aligned}$$

hence

$$R \varphi((n, q)) = \varphi(m)/\varphi(m_0) \quad (14)$$

Also  $\mu(q) = \mu((n, q)) \mu(q/(n, q))$ , so

$$\mu(q) \mu((n, q)) = \mu(q/(n, q)) = \mu(m_0/f). \quad (15)$$

Finally  $\alpha_0 = n/(n, q)$ , so  $\alpha_0 q = nm_0/f$ ,

hence

$$\psi(\alpha_0)\psi(q) = \psi(m_0/f)\psi(n)$$

or

$$\bar{\psi}(n)\psi(q) = \bar{\psi}(\alpha_0)\psi\left(\frac{m_0}{f}\right) \quad (16)$$

Multiplying (14), (15) and (16) we find  $K(\alpha) = H(\alpha)$ .

#### 4. SPECIAL CASES

(a) Theorem *B* implies that  $\tau(\chi) \neq 0$  if and only if  $R = 1$ , that is, if and only if  $m/f$  is squarefree and has no common divisor with  $f$ . We have then

$$\tau(\chi) = \mu\left(\frac{m}{f}\right)\psi\left(\frac{m}{f}\right)\tau(\psi) \quad (17)$$

and

$$\mathcal{G}(\alpha, \chi) = g(\alpha)\tau(\chi) \quad (18)$$

On the other hand, if  $m/f$  is not square free or has a common divisor with  $f$ , then the right hand side of (17) is zero. So, (17) holds for any character  $\chi$ . For another proof of this see [4], p. 148.

(b) If  $\chi = \chi_0$  = principal character mod  $m$ , then  $f = 1$ ,  $\psi \equiv 1$ ,  $\tau(\psi) = 1$ ,  $q = \tilde{m}$  = squarefree kernel of  $m$ ,  $R = m/\tilde{m}$ , and  $\mathcal{G}(\alpha, \chi_0) = C_m(\alpha)$  = RAMANUJANS SUM.

Theorem *B* gives the well-known formula:

$$C_m(\alpha) = 0 \quad \text{if} \quad \frac{m}{\tilde{m}} \not\mid \alpha$$

$$C_m\left(\frac{m}{\tilde{m}} n\right) = \frac{m}{\tilde{m}} \mu(\tilde{m}) \mu((n, \tilde{m})) \varphi((n, \tilde{m})).$$

From (17) we get for all  $m$

$$C_m(1) = \mu(m).$$

5. Remarks: (a) It is clear that  $\mathcal{G}(\alpha, \chi)$  cannot vanish identically. So by 4. (a), formula (1) can only hold if  $R = 1$ , and if  $g(\alpha) = \bar{\chi}(\alpha)$  for all  $\alpha$ . But this is only possible if  $q = 1$ , i.e. if  $m = f$ . This shows that (1) characterises primitive characters, a fact proved by T.M. Apostol [5].