4. Special cases

$$\psi(\alpha_0)\,\psi(q) = \psi(m_0/f)\,\psi(n)$$

or

$$\bar{\psi}(n)\,\psi(q) = \bar{\psi}(\alpha_0)\,\psi\left(\frac{m_0}{f}\right) \tag{16}$$

Multiplying (14), (15) and (16) we find $K(\alpha) = H(\alpha)$.

4. Special cases

(a) Theorem B implies that $\tau(\chi) \neq 0$ if and only if $R = 1$, that is, if and only if m/f is squarefree and has no common divisor with f. We have then

$$\tau(\chi) = \mu\left(\frac{m}{f}\right)\psi\left(\frac{m}{f}\right)\tau(\psi) \tag{17}$$

and

$$\mathcal{G}(\alpha, \chi) = g(\alpha)\,\tau(\chi) \tag{18}$$

On the other hand, if m/f is not square free or has a common divisor with f, then the right hand side of (17) is zero. So, (17) holds for any character χ. For another proof of this see [4], p. 148.

(b) If $\chi = \chi_0 =$ principal character mod m, then $f = 1$, $\psi \equiv 1$, $\tau(\psi) = 1$, $q = \tilde{m} =$ squarefree kernel of m, $R = m/\tilde{m}$, and $\mathcal{G}(\alpha, \chi_0) = C_m(\alpha) =$ Ramanujans sum.

Theorem B gives the well-known formula:

$$C_m(\alpha) = 0 \quad \text{if} \quad \frac{m}{\tilde{m}} \nmid \alpha$$

$$C_m\left(\frac{m}{\tilde{m}}n\right) = \frac{m}{\tilde{m}}\,\mu(\tilde{m})\,\mu((n, \tilde{m}))\,\varphi((n, \tilde{m}))\,.$$

From (17) we get for all m

$$C_m(1) = \mu(m)\,.$$

5. Remarks: (a) It is clear that $\mathcal{G}(\alpha, \chi)$ cannot vanish identically. So by 4. (a), formula (1) can only hold if $R = 1$, and if $g(\alpha) = \bar{\chi}(\alpha)$ for all α. But this is only possible if $q = 1$, i.e. if $m = f$. This shows that (1) characterises primitive characters, a fact proved by T.M. Apostol [5].

(b) The last named author also proved ([6]), that if the functional equation (4) holds, then χ is primitive. One may prove this by comparing (4) and (5), which gives $\mathcal{G}(\alpha, \chi) = \bar{\chi}(\alpha)\,\tau(\chi)$; this in turn implies that χ is primitive, by the former remark. Still another proof is as follows: If $q = 1$, then $L(s, \chi) = L(s, \psi)$ and $L(s, \bar{\chi}) = L(s, \bar{\psi})$. So if (4) holds, we get $m = f$, hence χ is primitive. If $q > 1$, then $L(s, \chi)$ must have nonreal zeroes on the imaginary axis; hence if (4) holds, $L(s, \bar{\chi})$ has zeroes on the line $\operatorname{Re} s = 1$, contradicting a well known theorem on L-series.

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H. Joris

Section de Mathématiques
Université de Genève
Case 124
1211 Genève 24