LINE BUNDLES ON THE MODULI SPACE

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **23 (1977)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **25.05.2024**

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$$\begin{array}{ccccc} C_\eta & \xleftarrow{\approx} & D_\eta & \xrightarrow{\Phi\eta} & \mathbf{P}^N \times \operatorname{Spec} k((t)) \\ \cap & & \cap & & \cap \\ \mathscr{C} & \longleftarrow & \mathscr{D} & \xrightarrow[\Phi]{} & \mathbf{P}^N \times \operatorname{Spec} k[[t]] \end{array}$$

$$\Phi_\eta^*(\mathcal{O}_{\mathbf{P}^N}(1)) = \omega_{D_\eta / k((t))}^{\otimes n}.$$

Let $L = \mathcal{O}_{\mathscr{D}}(1)$. It follows that $L \cong \omega_{\mathscr{D}/k[[t]]}^{\otimes n}(-\sum r_i D_i)$, where D_i are the components of D_0. Multiplying the isomorphism by $t^{\min(r_i)}$, we can assume $r_i \geqq 0$, $\min r_i = 0$. Let $D_1 = \bigcup_{r_i = 0} D_i$, $D_2 = \bigcup_{r_i > 0} D_i$. If f is a local equation of $\sum r_i D_i$, then $f \not\equiv 0$ in any component of D_1 since $r_i = 0$ on all these while $f(x) = 0$, all $x \in D_1 \cap D_2$, so

$$\#(D_1 \cap D_2) \leqq \deg_{D_1}(\mathcal{O}_{\mathscr{D}_0}(\textstyle\sum r_i D_i)).$$

But this last degree equals $(\deg D_1 - n \deg_{D_1}(\omega_{D_0}))$ which contradicts iii) of Proposition 5.5 unless all r_i are zero. Hence $L = \omega_{\mathscr{C}}^{\otimes n}$ which shows $\mathscr{D} = \mathscr{C}$.

Line bundles on the moduli space

For the remainder of this section we examine $\operatorname{Pic}(\overline{\mathscr{M}}_g)$. We fix a genus $g \geqq 2$ and an $e \geqq 3$. Then for all stable C, $\omega_C^{\otimes e}$ is very ample and in this embedding C has degree $d = 2e(g-1)$, the ambient space has dimension $\nu - 1$ where $\nu = (2e-1)(g-1)$ and C has Hilbert polynomial $P(X) = dX - (g-1)$. Let $H \subset \operatorname{Hilb}^P_{\mathbf{P}^{\nu-1}}$ be the locally closed smooth subscheme of e-canonical stable curves C, let $C \subset H \times \mathbf{P}^{\nu-1}$ be the universal curve and let

$$\mathrm{ch} : H \to \mathrm{Div} = \mathrm{Div}^{d,d} = \left\{ \begin{array}{l} \text{projective space of bihomogeneous forms} \\ \text{of bidegree } (d, d) \text{ in dual coordinates} \\ u, v \text{ (cf. § 1).} \end{array} \right\}$$

be the Chow map. These are related by the diagram

$$\begin{array}{ccccc} & & C & & \\ & & \big\downarrow{\scriptstyle \pi} & & \\ \mathrm{Div} & \xleftarrow{\mathrm{ch}} & H & \xrightarrow{\rho} & \overline{\mathscr{M}}_g = H/PGL(\nu) \end{array}$$

If $\operatorname{Pic}(H, PGL(\nu))$ is the Picard group of invertible sheaves on H with $PGL(\nu)$-action, we have a diagram

$$\operatorname{Pic}(\bar{\mathscr{M}}_g) \xrightarrow{\rho^*} \operatorname{Pic}(H, PGL(v)) \xrightarrow{\alpha} \operatorname{Pic}(H)^{PGL(v)} \subset \operatorname{Pic}(H).$$

In this situation, we have:

LEMMA 5.8. *In the sequence above, ρ^* is injective with torsion cokernel and α is an isomorphism.*

Proof. α is an isomorphism by Prop. 1.4 [14]; ρ^* injective is easy; coker ρ^* torsion can be proved, for instance, using Seshadri's construction, Th. 6.1 [19].
This lemma allows us to examine Pic $(\bar{\mathscr{M}}_g)$ by looking inside Pic $(H)^{PGL(v)}$ which is a much easier group to come to grips with.

DEFINITION 5.9. *Let $\Delta \subset H$ be the divisor of singular curves, $\delta = \mathcal{O}_H(\Delta)$ and $\lambda_n = \Lambda^{\max}(\pi_*(\omega_{C/H}{}^{\otimes n}))$, $(n \geq 1)$. We write λ for λ_1.*

The sheaves λ_n and δ are the most obviously interesting invertible sheaves on H from a moduli point of view. The next theorem expresses all of these in terms just involving λ and δ.

THEOREM 5.10. $\lambda_n = \mu^{\binom{n}{2}} \otimes \lambda$ *where* $\mu = \lambda^{12} \otimes \delta^{-1}$.

Proof. The proof is based on Grothendieck's relative Riemann-Roch theorem (see Borel-Serre [4]), which we will briefly recall.

Let X and Y be complete smooth varieties over k, $A(X)$ be the Chow ring of X and $\mathscr{F}$ be a coherent sheaf on X. Let $c_i(\mathscr{F}) \in A(X)$ denote the i^{th} Chern class of $\mathscr{F}$, Chern $(\mathscr{F}) \in A(X) \otimes \mathbf{Q}$ its Chern character and $\mathscr{T}(\mathscr{F}) \in A(X) \otimes \mathbf{Q}$ its Todd genus. These are related by:

$$\text{(5.11)}\quad \text{Chern}(\mathscr{F}) = rk\,\mathscr{F} + c_1(\mathscr{F}) + \frac{c_1(\mathscr{F})^2}{2} - c_2(\mathscr{F})$$

$$+ \text{ terms of higher codimension,}$$

$$\mathscr{T}(\mathscr{F}) = 1 - \frac{c_1(\mathscr{F})}{2} + \frac{c_1(\mathscr{F})^2 + c_2(\mathscr{F})}{12}$$

$$+ \text{ terms of higher codimension.}$$

Let $K(Y)$ be the Grothendieck group of $Y, f: X \to Y$ be a proper map, and $f_!(\mathscr{F}) = \sum (-1)^i [\mathbf{R}^i f_* \mathscr{F}] \in K(Y)$. The relative Riemann-Roch theorem expresses the Chern character of $f_!(\mathscr{F})$, *modulo torsion* as

$$\text{Chern}(f_!\,\mathscr{F}) = f_*(\text{Chern}\,\mathscr{F} \cdot \mathscr{T}(\Omega^1_{X/Y}))$$

which using (5.11) gives:

$$(5.12)\quad rkf_!\mathcal{F} + c_1(f_!\mathcal{F}) + \dots\dots\dots$$
$$= f_*\left[\left(rk(\mathcal{F}) + c_1(\mathcal{F}) + \frac{c_1(\mathcal{F})^2}{2} - c_2(\mathcal{F})\right)\cdot\right.$$
$$\left.\left(1 - \frac{c_1(\Omega^1_{X/Y})}{2} + \frac{c_1(\Omega^1_{X/Y})^2 + c_2(\Omega^1_{X/Y})}{12}\right)\right]$$

For the time being, we work implicitly modulo torsion.

Now suppose $\mathcal{F}$ is a line bundle such that $R^i f_*(\mathcal{F}) = 0$, $i > 0$ and suppose $\dim X = \dim Y + 1$. Then the codimension 1 term on the left of (5.12) (i.e. on Y) corresponds to the codimension two term on the right (i.e. on X). Since $c_2(\mathcal{F}) = 0$, this gives

$$(5.13)\quad c_1(f_*\mathcal{F}) = c_1(f_!\mathcal{F})$$
$$= f_*\left[\frac{c_1(\Omega^1_{X/Y})^2 + c_2(\Omega^1_{X/Y})}{12} - \frac{c_1(\mathcal{F})c_1(\Omega^1_{X/Y})}{2} + \frac{c_1(\mathcal{F})^2}{2}\right]$$

In case $f: C \to S$ is a moduli-stable curve over S, $X = C$ and $Y = S$, we can simplify this. Indeed I claim that if Sing C is the singular set on C and I_{sing} is its ideal, then

i) codim Sing $C = 2$

ii) the canonical homomorphism $\Omega^1_{C/S} \to \omega_{C/S}$ induces an isomorphism $\Omega^1_{C/S} = I_{\text{sing}} \cdot \omega_{C/S}$.

We certainly have the isomorphism of ii) off Sing C. At a singular point C has a local equation of the form $xy = t^n$, where t is a parameter on S, x and y are affine coordinates on the fibre. Moreover locally C is singular only at the points (0, 0) in the fibres where $t = 0$, so Sing C has codimension 2. Near the singular point

$$\Omega^1_{C/S} = (\mathcal{O}_C dx + \mathcal{O}_C dy)/(xdy + ydx)\,\mathcal{O}_C$$

while $\omega_{C/S}$ is the invertible sheaf generated by the differential ζ which is given by dx/x outside $x = 0$ and by $-dy/y$ outside $y = 0$. Thus

$$\Omega^1_{C/S} = \mathcal{M}_{(0,0),C}\cdot\zeta = \mathcal{M}_{(0,0),C}\cdot\omega_{C/S}\,.$$

Recall the following corollary to Riemann-Roch: if X is a smooth variety, $Y \subset X$ a subvariety of codim r and $\mathcal{F}$ is coherent on Y, then considering $\mathcal{F}$ as a sheaf on X

$$c_i(\mathscr{F}) = \begin{cases} 0, 1 \leqq i \leqq r-1 \\ \big((-1)^{r-1}(r-1)!\, rk\, \mathscr{F}\big)\, Y, i = r \end{cases}$$

Set $X = C$, $Y = \operatorname{Sing} C$ and $\mathscr{F} = \Omega^1_{C/S}$. The Whitney product formula applied to the chern classes of the exact sequence

$$0 \to \Omega^1_{C/S} \to \omega_{C/S} \to \omega_{C/S} \otimes \mathcal{O}_{\operatorname{Sing} C} \to 0$$

gives, taking account of the corollary

$$\begin{aligned} &1 + c_1(\omega_{C/S}) \\ &= (1 + c_1(\Omega^1_{C/S}) + c_2(\Omega^1_{C/S}) + \ldots..) \,.\, (1 + 0 - [\operatorname{Sing} C] + \ldots..) \end{aligned}$$

Equating terms of equal codimension, we see that $c_1(\Omega^1_{C/S}) = c_1(\omega)$ and $c_2(\Omega^1_{C/S}) = [\operatorname{Sing} C]$ so that (5.13) becomes

$$c_1(f_* \mathscr{F}) = f_* \left[\frac{c_1(\omega_{C/S})^2 + [\operatorname{Sing} C]}{12} - \frac{c_1(\mathscr{F})\, c_1(\omega_{C/S})}{2} + \frac{c_1(\mathscr{F})^2}{2} \right]$$

Applying this to the map $\pi \colon C \to H$, when $\mathscr{F} = \omega_{C/H}^{\otimes n}$ gives

$$\begin{aligned} \lambda_n &= \Lambda^{\max}(\pi_*\, \omega_{C/H}^{\otimes n}) = c_1(\pi_*\, \omega_{C/H}^{\otimes n}) \\ &= \pi_* \left[\frac{c_1(\omega_{C/H})^2 + [\operatorname{Sing} C]}{12} - \frac{c_1(\omega_{C/H}^{\otimes n})\, c_1(\omega_{C/H})}{2} + \frac{c_1(\omega_{C/H}^{\otimes n})^2}{2} \right] \\ &= \binom{n}{2} \pi_*\big(c_1(\omega_{C/H})^2\big) + \frac{\pi_*\big(c_1(\omega_{C/H})^2\big) + [\Delta]}{12} \end{aligned}$$

Setting[1] $n = 1$, we see that $\lambda = \left[\dfrac{\pi_*\big(c_1(\omega_{C/H})^2\big) + [\Delta]}{12} \right]$ and $\pi_*\big(c_1(\omega_{C/H})^2\big) = 12\lambda - [\Delta]$. Plugging these values back in gives us the theorem up to torsion. But in fact:

Lemma 5.14. *Over* $\mathbf{C}$, $\operatorname{Pic}\big(H, PGL(v)\big)$ *is torsion free.*

Note that this will prove what we want because the invertible sheaves that we are trying to show are isomorphic all "live" on the full scheme $H_{\mathbf{Z}}$ over $\operatorname{Spec} \mathbf{Z}$ of stable $\mathscr{C}$-canonical curves. If they are isomorphic on $H_{\mathbf{Z}}$, they are isomorphic after any base change. But on the other hand, I claim that $\operatorname{Pic}\big(H, PGL(v)\big)$ injects into $\operatorname{Pic}\big(H_{\mathbf{C}}, PGL_{\mathbf{C}}(v)\big)$:

---

[1]) For $n = 1$, $R^1\pi_*(\omega_{C/H})$ is not zero, but it is the trivial line bundle, hence doesn't affect $\pi_!$.

If L is a line bundle on H with $PGL(v)$ action such that $L \otimes \mathbf{C}$ is trivial over $H_{\mathbf{C}}$, then

$$\begin{array}{c} H^0(H, L)^{PGL(v)} \otimes \mathbf{C} = H^0(H_{\mathbf{C}}, L \otimes \mathbf{C})^{PGL(v)} \\ \wr \Big\downarrow \alpha \\ H^0(H_{\mathbf{C}}, \mathcal{O}_{H\mathbf{C}})^{PGL(v)} = \mathbf{C} \end{array}$$

since $H_{\mathbf{C}}/_{PGL(v)}$ is compact. Thus we can find a non-zero section $s \in H^0(H, L)^{PGL(v)}$, which over $\mathbf{C}$ can be used to give the trivialization α. Over $\mathbf{C}$, s has no zeros so the divisor $(s)_0$ of the zeros of s on H, has support only over the closed fibres of Spec $(\mathbf{Z})$. Mumford and Deligne [6] have shown that $H \to \operatorname{Spec} \mathbf{Z}$ is smooth with irreducible fibres, hence $(s)_0 = \sum r_i \pi^{-1}(p)$, $r_i \geqq 0$ i.e. $(s)_0 = (n)$ for some integer n. Then $\left(\frac{s}{n}\right)$ is a global section of L with no zeros so L is trivial.

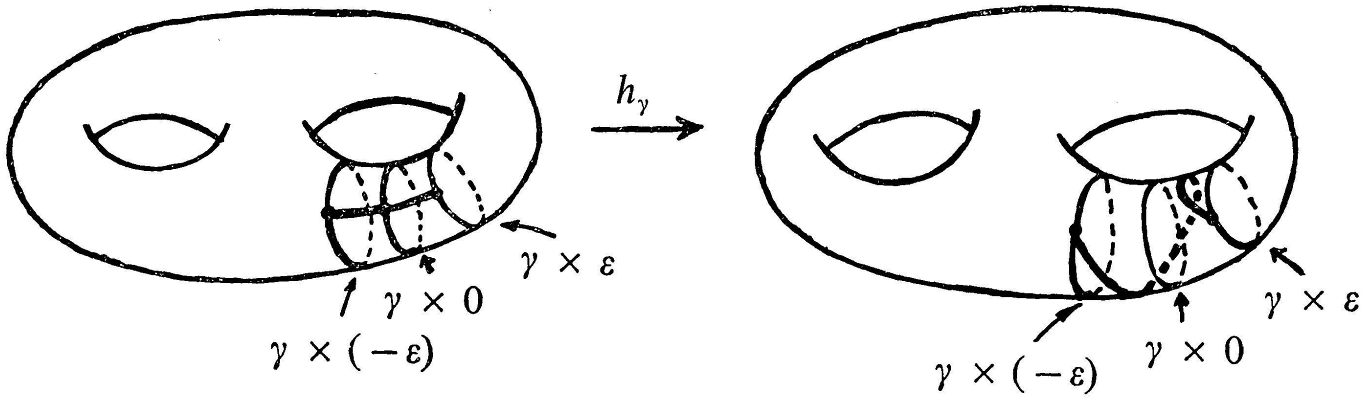
Proof of Lemma. Over $\mathbf{C}$, we have Teichmüller theory at our disposal. Let Π be a standard model of a group with generators $\{a_i, b_i \mid 1 \leqq i \leqq g\}$ mod the relation $\prod_{i=1}^{g} (a_i b_i a_i^{-1} b_i^{-1}) = 1$. Then the Teichmüller modular group Γ is

$$\Gamma = \{\alpha \mid \alpha : \Pi \to \Pi \text{ is an orientation preserving isomorphism}\} / \text{inner automorphisms}$$

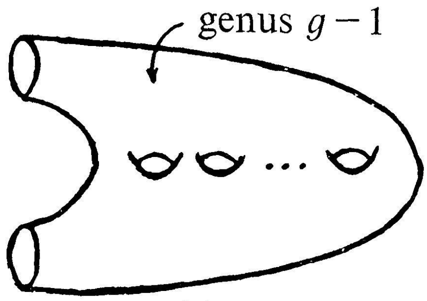
The Teichmüller space $\mathcal{T}_g$ is given by

$$\mathcal{T}_g = \left\{ (C, \alpha) \;\middle|\; \begin{array}{l} C \text{ a smooth curve of genus } g \text{ and } \alpha : \pi_1(C) \to \Pi \text{ an} \\ \text{orientation preserving isomorphism given up to inner} \\ \text{automorphism} \end{array} \right\}$$

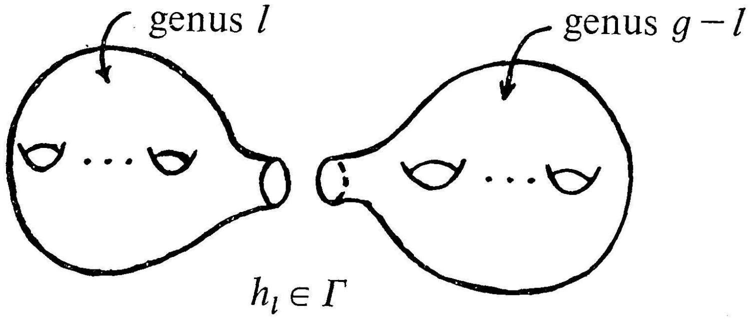
Fix a model M_g of the real surface of genus g, and identify $\pi_1(M_g)$ and Π. Then Γ is generated by the maps which are induced by certain automorphisms of M_g, called Dehn twists. The Dehn twist h_γ corresponding to a loop γ: $[0, 1] \to M_g$ on M_g is given by taking an ε-collar $\gamma \times [-\varepsilon, \varepsilon]$ about γ, letting h = identify off the collar and letting $h(\gamma(t), \eta - \varepsilon) = \left(\gamma\left(t + \frac{\eta}{2\varepsilon}\right), \eta - \varepsilon\right)$ as shown below.



Up to inner automorphism h_γ is determined by which of the pictures below results from cutting open M_g along γ. We have name these elements of Γ in the diagrams:



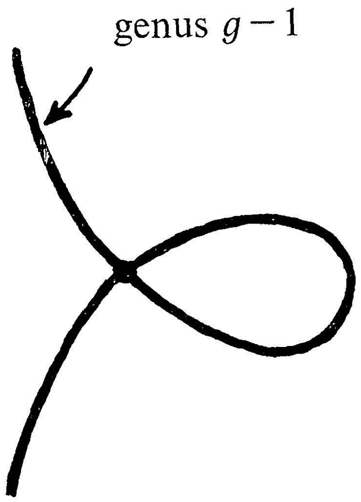
$h' \in \Gamma$

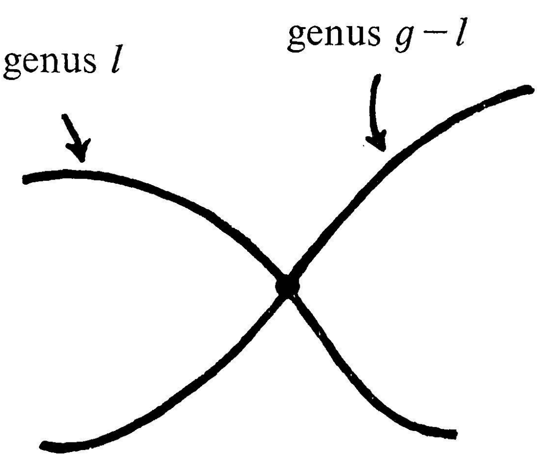


$h_l \in \Gamma$

The Dehn twist h_γ can also be described as the monodromy map obtained by going around a curve C_0 with one double point for which γ is the vanishing cycle.

The components of $\Delta \subset H$ correspond to the different ways of putting a stable double point on a smooth moduli stable curve C. They are the closures of the sets of curves of the forms shown below: again, we name these components in the diagram:





$$\Delta' = \begin{pmatrix}\text{closure of set} \\ \text{formed by curves} \\ \text{like this}\end{pmatrix} \qquad \Delta_l = \begin{pmatrix}\text{closure of set} \\ \text{formed by curves} \\ \text{like this}\end{pmatrix}$$

Let $\tilde{H} = \left\{ (C, \alpha, B) \;\middle|\; \begin{array}{l} (C, \alpha) \in \mathscr{T}_g, B \text{ a basis of the } e\text{-tuple dif-} \\ \text{ferentials on } C \text{ given up to a scalar} \end{array} \right\}$

Suppose we are given a line bundle L on H with $PGL(\nu)$-action such that $L^n \cong \mathcal{O}_H$. L induces a cyclic covering H' of H plus a lifting of the $PGL(\nu)$-action to H'. If we choose n minimal this covering is not split: we denote its structure group by Γ_L. Let $\tilde{H}'$ be the pullback of covering over $\tilde{H}$, and let $\mathscr{T}'_g$ denote the quotient of $\tilde{H}'$ by $PGL(\nu)$—this is a covering of $\mathscr{T}_g$. These coverings are related by

$$\begin{tikzcd}
 & & & \tilde{H}^1 \arrow[dl] \arrow[dd, no head] \arrow[dr] & \\
H' \arrow[dd, "\Gamma_L"'] & \supset & H' - \Delta' \arrow[dd, "\Gamma_L"'] & & \mathscr{T}'_g \arrow[dd] \\
 & & & \tilde{H} \arrow[ul, dashed] \arrow[dl, "\Gamma"] \arrow[dr] & \\
H & \supset & H - \Delta & & \mathscr{T}_g
\end{tikzcd}$$

$\mathscr{T}_g$ is simply connected so the cover $\mathscr{T}'_g \to \mathscr{T}_g$ splits, hence so does $\tilde{H}' \to \tilde{H}$. A section of this last cover gives a map from $\tilde{H}$ to $H' - \Delta'$ (shown dashed in the diagram), so Γ_L is a quotient of Γ, of finite order.

Let γ' [resp. γ_e] be a loop at a fixed base point $P_0 \in H - \Delta$ going around Δ' [resp.: Δ_e] but homotopic to 0 in H. Fix a point $\tilde{P}_0 \in \tilde{H}$ over P_0. The monodromy characterization of the Dehn twists implies that γ' [resp.: γ_e] lifted to $\tilde{H}$ goes from $\tilde{P}_0$ to $h'(\tilde{P}_0)$ [resp.: to $h_e(\tilde{P}_0)$]. Since γ' [resp.: γ_e] are homotopic to 0 in H, and the covering $H' - \Delta'$ extends over H, this implies that the image of h' [resp.: h_l] in Γ_L is 0. But these elements and their conjugates generate Γ_L, so $\Gamma_L = \{1\}$, hence $L \cong \mathcal{O}_H$, proving the lemma and the theorem.

In order to describe the ample cone on Pic $(\overline{\mathscr{M}_g})$ we prove:

THEOREM 5.15. $\mathrm{Ch}^*(\mathcal{O}_{\mathrm{Div}}(\nu)) = (\mu^e \otimes \lambda^{-4})^{e(g-1)}$

Proof. The proof depends on a result which we simply quote from Fogarty [8] or Knudsen [12]:

PROPOSITION 5.16. *Let* S *be a locally closed subscheme of a Hilbert scheme* $\mathrm{Hilb}^{P}_{\mathbf{P}^{\nu-1}}$, Ch *be the associated Chow map* $\mathrm{Ch}: S \to \mathrm{Div}$ *and* $Z \subset \mathbf{P}^{\nu} \times S$ *have relative dimension r over S. Then if* $n \gg 0$, $\Lambda^{\max} p_{2,*}(\mathcal{O}_Z(n)) = \bigotimes_{i=0}^{r+1} \mu_i^{\binom{n}{i}}$ *and* $\mathrm{Ch}^*(\mathcal{O}_{\mathrm{Div}}(1)) = \mu_{r+1}$, *where* μ_i *are suitable invertible sheaves on* S.

In the situation of our theorem, with $S = H$ and $Z = C$, $\mathcal{O}_C(1) = \omega_{C/H}^{\otimes e} \otimes \pi^* Q$ where Q is the invertible sheaf determined by $(\pi_* \omega_{C/H}^{\otimes e}) \otimes Q = \pi_* \mathcal{O}_C(1) = \pi_* \mathcal{O}_{\mathbf{P}^{\nu-1}}(1) = \mathcal{O}_H^{\nu}$, hence

$$\mathcal{O}_H = [\Lambda^{\max} \pi_* (\omega_{C/H}^{\otimes e})] \otimes Q^{\nu} = \mu^{\binom{e}{2}} \otimes \lambda \otimes Q^{\nu}. \tag{5.17}$$

On the other hand,

$$\Lambda^{\max}(\pi_* \mathcal{O}_C(n)) = \Lambda^{\max}[\pi_* (\omega_{C/H}^{\otimes ne}) \otimes Q^n] = \mu^{\binom{ne}{2}} \otimes \lambda \otimes Q^{P(n).n}.$$

This has leading term in n of $\mu^{n^2 e^2/2} \otimes Q^{2e(g-1)n^2}$ so

$$\begin{aligned} Ch^*(\mathcal{O}_{\mathrm{Div}}(\nu)) &= \mu^{\nu e^2} \otimes Q^{4e(g-1)\nu} \\ &= \mu^{\nu e^2 - \binom{e}{2}.4e(g-1)} \otimes \lambda^{-4e(g-1)} \qquad \text{using (5.17)}. \end{aligned}$$

Finally, therefore, $\mathrm{Ch}^*(\mathcal{O}_{\mathrm{Div}}(\nu)) = \mu^{e^2(g-1)} \otimes \lambda^{-4e(g-1)}$ as required.

COROLLARY 5.18. *If* $e \geqq 5$, $\mu^e \otimes \lambda^{-4}$ $(=\lambda^{12e-4} \otimes \delta^{-e})$ *is "ample on* $\overline{\mathcal{M}}_g$", *i.e. those positive powers of this bundle which are pull-backs of bundles on* $\overline{\mathcal{M}}_g$ *are ample on* $\overline{\mathcal{M}}_g$.

Proof. This is an immediate consequence of the Theorem and our main result: that $PGL(\nu)$-invariant sections of $\mathrm{Ch}^*(\mathcal{O}_{\mathrm{Div}}(1))$ define a projective embedding of $\overline{\mathcal{M}}_g$.

REMARK 5.19. A similar argument using the facts that

(1) $\omega^{\otimes e}$ is base point free for all canonical curves when $e \geqq 2$,

(2) smooth curves are stable if $d > 2g$,

shows that if $e \geqq 2$, the sections of $\lambda^{12e-4} \otimes \delta^{-e}$ on $\overline{\mathcal{M}}_g$ separate points on $\mathcal{M}_g$.

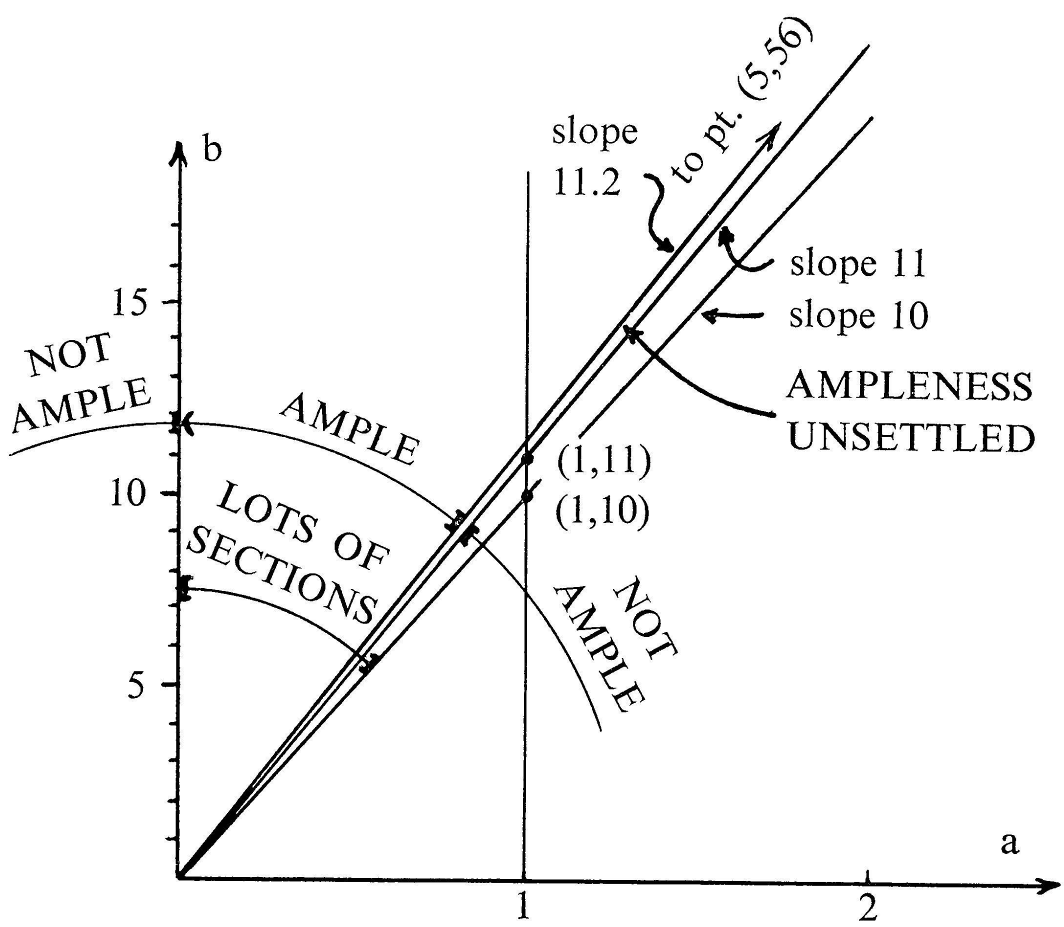
To get a good picture of the ample cone on $\overline{\mathcal{M}}_g$ we need to use the realization via Θ functions $\mathcal{A}_{g,1} \xrightarrow{\Theta} \mathbf{P}^N$ of the moduli scheme $\mathcal{A}_{g,1}$ of principally polarized abelian varieties. More precisely, let $J : \mathscr{M}_g \to \mathscr{A}_{g,1}$ be the map taking a curve C to its Jacobian. Then we have:

THEOREM 5.20. *In characteristic* 0, *the morphism* $\mathscr{M}_g \xrightarrow{J} \mathscr{A}_{g,1} \xrightarrow{\Theta} \mathbf{P}^N$ *extends to a morphism* $\overline{\mathscr{M}_g} \xrightarrow{\theta} \mathbf{P}^N$ *so that for some* m, $\theta^*(\mathcal{O}_{\mathbf{P}^N}(1)) = \lambda^m$.

Proof. See Arakelov [1] or Knudsen [12].

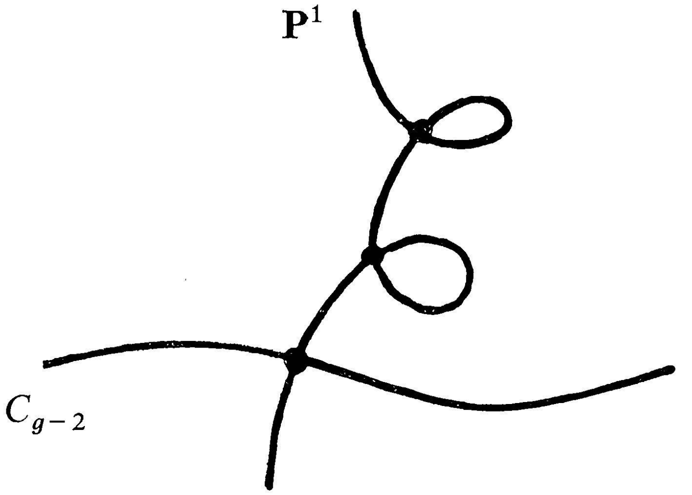
REMARK. This should also hold in characteristic p, but it seems to be a rather messy problem there.

Putting together 5.18 and 5.20, we get a whole sector in the (a, b)-plane such that $\lambda^b \otimes \delta^{-a}$ is ample for (a, b) in this sector. This is depicted in the diagram below:



The fact that λ and $\lambda^{11} \otimes \delta^{-1}$ are not ample can be seen by examining the following 2 curves in $\overline{\mathscr{M}_g}$:

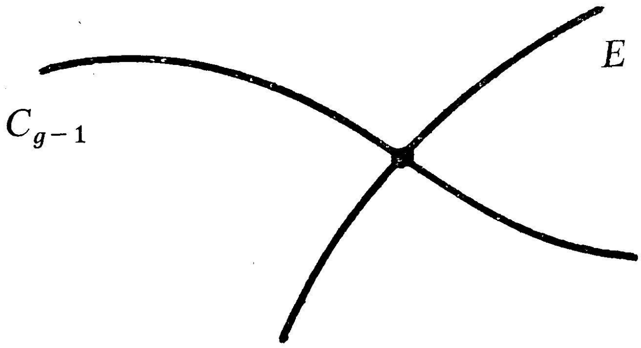
(1) If S_1 is a curve in $\overline{\mathscr{M}_g}$ composed of curves of the form:



$\left\{\begin{array}{l}\text{vary cross ratio of the 4 points}\\ \text{in } \mathbf{P}^1 \text{ making up the 2 double points}\end{array}\right.$

where C_{g-2} is a fixed genus $(g-2)$ component, then $\lambda \mid_{S_1} = \mathcal{O}_{S_1}$, hence sections of λ always collapse such families.

(2) If S_2 is a curve in $\overline{\mathcal{M}}_g$ composed of curves of the form:



$\left\{\begin{array}{l}E \text{ an elliptic curve: vary its}\\ j\text{-invariant}\end{array}\right.$

where C_{g-1} is a fixed genus $(g-1)$ component, then $\lambda^{11} \otimes \delta^{-1} \mid_{S_2} = \mathcal{O}_{S_2}$ i.e. $\lambda^{11} \otimes \delta^{-1}$ collapses these families.

We omit the details.

APPENDIX

We wish to fill in the gap in the proof of Proposition 5.5 on page 95. The difficulty occurs if the support of $\mathcal{I}$, i.e. $(0) \times L_1$, contains some of the components of C_2 meeting C_1. In this case, the inequality

$$e_L(\mathcal{I}_2) \geqslant w$$

is not clear. Indeed, if $D_1, \ldots, D_k$ are the components of C_2 meeting C_1, $w_i = \#(D_i \cap C_1)$, and $\mathcal{K}_i$ is the pull-back of $\mathcal{I}_2$ to D_i, then