3. Historical Development

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Are there examples of B*-algebras other than the above? Numerous mathematical papers have been devoted to answering this question. In the remainder of this article we shall be occupied not only with its history and solution, but also with recent developments which have been stimulated by it.

3. Historical development

In 1943 the Soviet mathematicians Gelfand and Naimark published (in English!) a ground-breaking paper [23] in which they proved that a Banach *-algebra with an identity element e is isometrically *-isomorphic to a C*-algebra if it satisfies the following three conditions:

1°	$\| x^*x \| = \| x^* \| \cdot \| x \|$	(the B*-condition);
2°	$\| x^* \| = \| x \|$	(isometric involution);
3°	e + x^*x is invertible	(symmetry)

for all x. They immediately asked in a footnote if conditions 2° and 3° could be deleted—apparently recognizing that they were of a different character than condition 1° and were needed primarily because of their method of proof. This indeed turned out to be true after considerable work. To trace the resulting history in detail it is convenient to look at the commutative and noncommutative cases separately.

Commutative algebras: In their paper Gelfand and Naimark first proved that every commutative B*-algebra with identity is a $C(X)$ for some compact Hausdorff space X. In the presence of commutativity they were able to show quite simply that the B*-condition implies the involution is isometric. Utilizing a delicate argument depending on the notion of "Shilov boundary" they proved that every commutative B*-algebra is symmetric. Thus in the commutative case they were able to show that conditions 2° and 3° follow from condition 1°.

A much simpler proof for the symmetry of a commutative B*-algebra was published in 1946 by Richard Arens [3]. It may be of some historical interest to mention that Professor Arens—as he pointed out to the first named author during a conversation—had not seen Gelfand-Naimark's proof when he found his. In 1952, utilizing the exponential function for elements of a Banach algebra, the Japanese mathematician Masanori Fukamiya published [21] yet another beautiful proof of symmetry. These arguments of Arens and Fukamiya will be given in full in the next section.

Noncommutative algebras: The 1952 paper of Fukamiya [21] implicitly contained the key lemma needed to eliminate condition 3° for noncommutative algebras. In essence this lemma states that if x and y are "positive" elements in a B*-algebra with identity and isometric involution, then $x + y$ is also positive. Independently and nearly simultaneously this lemma was discovered by John L. Kelley and Robert L. Vaught [31]. The Kelley-Vaught argument is extremely brief and elegant, and is the one that we shall give in Section 5.

The nontrivial observation that this lemma was the key to eliminating condition 3° was due to Irving Kaplansky. His ingenious argument was recorded in Joseph A. Schatz's review [45] of Fukamiya's paper, making it an amusing instance where a theorem was first "proved" in the Mathematical Reviews.

In marked contrast to the commutative case, the redundancy of condition 2° for noncommutative algebras did not follow easily; in fact, the question remained open until 1960 when a solution for B*-algebras with identity was published by James G. Glimm and Richard V. Kadison [25]. Their proof was based on a deep "n-fold transitivity" theorem for unitary operators in an irreducible C*-algebra. A beautiful theorem of Bernard Russo and Henry A. Dye [44] made it possible to by-pass the Glimm-Kadison transitivity theorem; an elementary proof of their result was given recently by Lawrence A. Harris [28]. We mention that another paper concerning the elimination of 2° (and also 3°) was published by the Japanese mathematician Tamio Ono [39] in 1959. However this paper appeared to have errors in the arguments of both the main theorems (see the review of [39]). Ten years later Ono [40] acknowledged these mistakes and corrected them from the viewpoint of 1959.

The original conjecture of Gelfand and Naimark was, at this time, completely solved for algebras with identity. What about algebras without identity? This question is of considerable importance since most C*-algebras which occur in applications do not possess an identity. An answer was provided in 1967 by B. J. Vowden [54]. He was able to utilize the notion of "approximate identity" and several arguments from Ono [39] to embed a B*-algebra without identity in a B*-algebra with an identity. He then applied the case for algebras with identity to complete the proof. Hence after nearly twenty five years of work the mathematical community had the theorems as we have stated them in the introduction.