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REPRESENTATION OF COMPLETELY CONVEX FUNCTIONS BY THE EXTREME-POINT METHOD

by CHRISTIAN BERG

0. INTRODUCTION

A function $f:]0, 1[\rightarrow \mathbf{R}$ is called completely convex, if it is C^∞ and $(-1)^k f^{(2k)} \geq 0$ for all $k \geq 0$. A completely convex function f is called minimal if $f(x) - a \sin(\pi x)$ is not completely convex for any number $a > 0$. Widder showed (cf. [5]) that a completely convex function can be extended to an entire holomorphic function, and in the paper [6] he proved that a minimal completely convex function can be expanded in a Lidstone series. This indicates that the Lidstone polynomials lie on the extreme rays of the cone W of completely convex functions.

The purpose of the present paper is to treat the completely convex functions by the extreme-point method and to obtain the expansion in Lidstone series as a special case of the Choquet representation theorem.

We will proceed as follows: In the topology of point-wise convergence the set W of completely convex functions is a closed, metrizable convex cone. We prove directly that the extreme rays of W are generated by certain polynomials — essentially the Lidstone polynomials — and the function $\sin(\pi x)$. The occurrence of the extreme ray generated by $\sin(\pi x)$ is related to the fact that only minimal completely convex functions can be expanded in Lidstone series.

The cone W has a compact convex base B , and the extreme points of B are determined. It turns out that B is a Bauer simplex, i.e. B is a simplex and the extreme points form a closed set.

The author wants to acknowledge Widder's paper [6] as a source of inspiration. The reason for writing this paper is that we felt it natural to investigate the cone W by the extreme-point method.

Recently Mugler [2] showed that real part of the holomorphic extension of $f \in W$ to the strip $\operatorname{Re} z \in]0, 1[$ is completely convex on each segment $\{x + iy \mid 0 < x < 1\}$. We give a very short proof of this result.

1. COMPLETELY CONVEX FUNCTIONS

Let I denote an open interval. A function $f : I \rightarrow \mathbf{R}$ is called *completely convex*, if it is C^∞ and $(-1)^k f^{(2k)} \geq 0$ on I for $k \geq 0$.

The set of completely convex functions is a convex cone denoted $W = W(I)$. We always equip W with the topology of pointwise convergence, i.e., with the topology induced by the product space \mathbf{R}^I .

LEMMA 1.1. *If I is unbounded $W(I)$ consists of the non-negative affine functions, and $W(\mathbf{R})$ consists of the non-negative constants.*

Proof. Assume first that $\inf I = -\infty$. Then every $f \in W$ is decreasing since it is non-negative and concave. For $k \geq 0$ and $f \in W$ we have $(-1)^k f^{(2k)} \in W$ and consequently $(-1)^k f^{(2k+1)} \leq 0$. This shows that also $-f' \in W$ and then $-f'' \leq 0$, but by definition $f'' \leq 0$ and therefore f is affine.

The case $\sup I = \infty$ is treated in a similar manner. Finally, every non-negative concave function on \mathbf{R} is constant.

Remark. Completely convex sequences are non-negative and affine.

For a sequence $a = (a_0, a_1, \dots)$ of real numbers we define Δa to be the sequence $(\Delta a)_n = a_{n+1} - a_n, n \geq 0$, and $\Delta^k a$ is defined as $\Delta(\Delta^{k-1} a)$ for $k \geq 1$, where $\Delta^0 a = a$. A sequence a is called *completely convex* if $(-1)^k \Delta^{2k} a \geq 0$ for $k \geq 0$. The same method as in Lemma 1.1 leads to the conclusion that every completely convex sequence satisfies $\Delta a \geq 0$ and $\Delta^2 a = 0$. The completely convex sequences are therefore exactly the sequences $a_n = \alpha n + \beta$, where $\alpha, \beta \geq 0$.

This is an answer to a remark by Boas [1]: "Nothing seems to be known about completely convex sequences".

In the following we will always assume that I is bounded, and for the sake of convenience we choose I to be $I =]0, 1[$. We simply write W for $W(]0, 1[)$. For $f \in W$ we have $-f'' \in W$ and $f^* \in W$, where f^* is defined by $f^*(x) = f(1-x)$. The mapping $f \mapsto f^*$ is an affine isomorphism of W onto itself.

LEMMA 1.2. *Let $f :]0, 1[\rightarrow \mathbf{R}$ be non-negative and concave. Then the following holds:*

$$(i) \quad f(x) \leq 2f(1/2) \quad \text{for} \quad x \in]0, 1[.$$

$$(ii) \quad f(x) \geq \frac{1}{\pi} f(x_0) \sin(\pi x) \quad \text{for} \quad x, x_0 \in]0, 1[.$$

(iii) ([6], Lemma 7.1) *If there exists $x_0 \in]0, 1[$ and $a > 0$ such that $f(x_0) < a \sin(\pi x_0)$ then $f(x) \leq a\pi$ for $x \in]0, 1[$.*

Proof. (i). For $x \in]0, 1/2]$ we have that $f(x)$ lies below the line through $(1/2, f(1/2))$ and $(1, 0)$ and (i) follows for $x \in]0, 1/2]$. The interval $[1/2, 1[$ is treated similarly.

(ii). Let $x_0 \in]0, 1[$. For $x \in]0, x_0]$ we have

$$f(x) \geq \frac{f(x_0)}{x_0} x \geq f(x_0) x \geq \frac{f(x_0)}{\pi} \sin(\pi x),$$

and for $x \in [x_0, 1[$ we have

$$\begin{aligned} f(x) &\geq \frac{f(x_0)(1-x)}{1-x_0} \geq f(x_0)(1-x) \geq \frac{f(x_0)}{\pi} \sin \pi(1-x) = \\ &\frac{f(x_0)}{\pi} \sin(\pi x). \end{aligned}$$

(iii). If $f(x_0) > a\pi$ the inequality (ii) implies that $f(x) > a \sin(\pi x)$ for $x \in]0, 1[$.

Since every $f \in W$ can be extended to an entire holomorphic function all derivatives of f have finite limits at 0 and 1. This can also be established in an elementary way from the property $(-1)^k f^{(2k)} \geq 0$ for $k \geq 0$. We will therefore freely use $f^{(k)}(x)$ for $x = 0, 1$ as the limit of $f^{(k)}(x)$ at these points.

LEMMA 1.3. *The cone W is a closed and metrizable subset of \mathbf{R}^I .*

Proof. The set of concave functions $f: I \rightarrow \mathbf{R}$ is a closed and metrizable subset of \mathbf{R}^I , and therefore it suffices to prove that the pointwise limit f of a sequence (f_n) from W belongs to W .

It follows by Lemma 1.2 (i) that there exists a constant A such that $f_n \leq A$ for all n ¹⁾. The dominated convergence theorem then shows that

$$\lim_{n \rightarrow \infty} \int_0^1 f_n(x) \varphi(x) dx = \int_0^1 f(x) \varphi(x) dx$$

for all $\varphi \in \mathcal{D}(]0, 1[)$, so (f_n) converges to f weakly in the distribution sense. Therefore $(-1)^k f^{(2k)} \geq 0$ for all $k \geq 0$ in the distribution sense, and this implies that f is C^∞ and hence $f \in W$.

¹⁾ In fact, $A = 2 \sup f_n(1/2)$ can be used. It is finite because $\lim_{n \rightarrow \infty} f_n(1/2)$ exists.

2. DETERMINATION OF THE EXTREME RAYS OF W

Inspired by [6] we consider the Green's function

$$G(x, t) = \begin{cases} (1-x)t & \text{for } 0 \leq t < x \leq 1, \\ (1-t)x & \text{for } 0 \leq x \leq t \leq 1. \end{cases}$$

If φ is a continuous function on $[0, 1]$ the unique solution $f \in C([0, 1]) \cap C^2(]0, 1[)$ to the equations

$$(2.1) \quad f'' = -\varphi \text{ in }]0, 1[, \quad f(0) = f(1) = 0$$

is

$$(2.2) \quad f(x) = \int_0^1 G(x, t) \varphi(t) dt.$$

The successive iterates of G are defined for $x, t \in [0, 1]$ by the equations

$$G_1(x, t) = G(x, t)$$

$$G_n(x, t) = \int_0^1 G(x, y) G_{n-1}(y, t) dy, \quad n \geq 2.$$

It is clear that $G_n(x, t) \geq 0$ for $x, t \in [0, 1]$.

We define recursively a sequence of polynomials $(A_n)_{n \geq 0}^1$ by the requirement

$$(2.3) \quad A_0(x) = x, \quad A_n'' = -A_{n-1} \quad \text{and} \quad A_n(0) = A_n(1) = 0 \\ \text{for } n \geq 1.$$

The polynomial A_n is of degree $(2n + 1)$, and we clearly have

$$(2.4) \quad A_n(x) = \int_0^1 G(x, t) A_{n-1}(t) dt = \int_0^1 G_n(x, t) t dt \quad \text{for} \\ n \geq 1, \quad x \in [0, 1].$$

It follows that $A_n \geq 0$ on $[0, 1]$ for all n , and since $(-1)^k A_n^{(2k)} = A_{n-k}$ for $k \leq n$ we see that $A_n \in W$.

We recall that a ray \mathbf{R}_+x of a cone C is called *extreme*, if an equation $x = f + g$ with $f, g \in C$ is possible only if $f, g \in \mathbf{R}_+x$, cf. [3].

¹) Our terminology is different from that of [6]; $(-1)^n A_n$ is equal to the n 'th Lidstone polynomial of [4] and [6].

PROPOSITION 2.1. *The polynomials $A_n, n \geq 0$, lie on extreme rays of W .*

Proof. If $A_0 = f + g$ with $f, g \in W$ we have $0 = f'' + g''$, but since f'' and g'' are both ≤ 0 , we conclude that f and g are affine. Furthermore, since $f(0) = g(0) = 0$, we conclude that f and g are proportional to A_0 .

Suppose now that $A_{n-1}, n \geq 1$, lies on an extreme ray of W , and assume that $A_n = f + g$ where $f, g \in W$. Then $A_{n-1} = -f'' + (-g'')$, and the induction hypothesis implies that $-f''$ and $-g''$ are proportional to A_{n-1} . Therefore we have $f = \lambda A_n(x) + ax + b$ for certain numbers $\lambda \geq 0, a, b$. Since $0 \leq f \leq A_n$, we have $f(0) = f(1) = 0$ which implies that $a = b = 0$. This proves that f (and similarly g) are proportional to A_n which then lies on an extreme ray of W .

Since $f \mapsto f^*$ is an affine isomorphism of W the polynomials A_n^* also lie on extreme rays of W . The following result is a special case of [6], Theorem 1.1.

PROPOSITION 2.2. *Every function $f \in W$ can for $n \geq 1$ be written as*

$$f(x) = \sum_{k=0}^{n-1} ((-1)^k f^{(2k)}(0) A_k^*(x) + (-1)^k f^{(2k)}(1) A_k(x)) + R_n(x),$$

where

$$R_n(x) = \int_0^1 G_n(x, t) (-1)^n f^{(2n)}(t) dt \in W.$$

Proof. For $n = 1$ the formula is equivalent with

$$(2.5) \quad f(x) - f(0)(1-x) - f(1)x = R_1(x) = - \int_0^1 G(x, t) f''(t) dt,$$

which follows directly from (2.2), and it is clear that $R_1 \in W$.

Suppose now the formula holds for some $n \geq 1$. Applying (2.5) to $(-1)^n f^{(2n)} \in W$ we get

$$\begin{aligned} (-1)^n f^{(2n)}(x) &= (-1)^n f^{(2n)}(0) A_0^*(x) + (-1)^n f^{(2n)}(1) A_0(x) \\ &+ \int_0^1 G(x, t) (-1)^{n+1} f^{(2n+2)}(t) dt, \end{aligned}$$

which substituted in the expression for R_n yields the formula for $n+1$ because of (2.4).

To see that $R_n \in W$ we notice that

$$(-1)^k R_n^{(2k)}(x) = \begin{cases} \int_0^1 G_{n-k}(x, t) (-1)^n f^{(2n)}(t) dt & \text{for } 0 \leq k \leq n-1 \\ (-1)^k f^{(2k)}(x) & \text{for } k \geq n. \end{cases}$$

The following lemma is easy to establish, but instead of giving the proof here we refer to [6].

LEMMA 2.3. *There exists a constant $M > 0$ such that*

$$0 \leq \int_0^1 G_n(x, t) dt \leq \frac{M}{\pi^{2n}} \quad \text{for} \quad 0 \leq x \leq 1, \quad n \geq 1.$$

PROPOSITION 2.4. *The only functions $f \in W$ satisfying $f^{(2k)}(0) = f^{(2k)}(1) = 0$ for all $k \geq 0$ are $f(x) = a \sin(\pi x)$ with $a \geq 0$.*

Proof. Suppose $f \in W$ satisfies $f^{(2k)}(0) = f^{(2k)}(1) = 0$ for all $k \geq 0$. Defining $a = \sup \{ \alpha \geq 0 \mid f - \alpha \sin(\pi x) \in W \}$, $g = f - a \sin(\pi x)$ belongs to W because W is closed in \mathbf{R}^I . Furthermore

$$g^{(2k)}(0) = g^{(2k)}(1) = 0 \quad \text{for all } k \geq 0.$$

Let $\varepsilon > 0$ be given. Since $\varphi = g - \varepsilon \sin(\pi x) \notin W$, there exist $k \geq 0$ and $x_0 \in]0, 1[$ such that $(-1)^k \varphi^{(2k)}(x_0) < 0$, hence

$$(-1)^k g^{(2k)}(x_0) < \varepsilon \pi^{2k} \sin(\pi x_0).$$

By Lemma 1.2 (iii) applied to $(-1)^k g^{(2k)}$ we get

$$(-1)^k g^{(2k)}(t) \leq \varepsilon \pi^{2k+1} \quad \text{for} \quad 0 < t < 1,$$

and therefore by Proposition 2.2 and Lemma 2.3 for $0 < x < 1$

$$\begin{aligned} g(x) &= \int_0^1 G_k(x, t) (-1)^k g^{(2k)}(t) dt \leq \varepsilon \pi^{2k+1} \int_0^1 G_k(x, t) dt \\ &\leq \varepsilon M \pi. \end{aligned}$$

This proves that g is identically zero.

PROPOSITION 2.5. *The extreme rays of W are precisely the rays generated by Λ_n and Λ_n^* , where $n \geq 0$, and $\sin(\pi x)$.*

Proof. We first show that $\sin(\pi x)$ lies on an extreme ray. If $\sin(\pi x) = f + g$ where $f, g \in W$, we have $f(0) = f(1) = g(0) = g(1) = 0$. Differentiating $2k$ times we similarly get $f^{(2k)}(0) = f^{(2k)}(1) = g^{(2k)}(0) = g^{(2k)}(1) = 0$, and it follows by Proposition 2.4 that f and g are proportional to $\sin(\pi x)$.

We finally have to show that an arbitrary extreme ray is generated by one of the above functions.

Assume that $f \in W$ generates an extreme ray. If $f^{(2k)}(0) = f^{(2k)}(1) = 0$ for all $k \geq 0$ we already know by Proposition 2.4 that f is proportional to $\sin(\pi x)$. Otherwise let n be the smallest number ≥ 0 for which $f^{(2n)}(0) \neq 0$ or $f^{(2n)}(1) \neq 0$. By Proposition 2.2 we then have

$$f(x) = (-1)^n f^{(2n)}(0) A_n^*(x) + (-1)^n f^{(2n)}(1) A_n(x) + R_{n+1}(x),$$

but since f lies on an extreme ray all three terms on the right-hand side lie on this ray.

If $f^{(2n)}(0) \neq 0$ this shows that $(-1)^n f^{(2n)}(1) A_n$ and R_{n+1} are proportional to A_n^* . Therefore $f^{(2n)}(1) = 0$ and $R_{n+1}^{(2n+2)} = f^{(2n+2)}$ is proportional to $(A_n^*)^{(2n+2)} = 0$, so that $f^{(2n+2)} = 0$ and hence $R_{n+1} = 0$ (cf. Proposition 2.2).

If $f^{(2n)}(1) \neq 0$ we similarly get $f^{(2n)}(0) = 0$ and $R_{n+1} = 0$. This shows that f lies on the ray generated by either A_n^* or A_n .

3. DETERMINATION OF A BASE FOR W

There are several ways of determining a base for W . We choose the following set

$$B = \left\{ f \in W \mid \int_0^1 f(x) \sin(\pi x) dx = 1 \right\}.$$

By Lemma 1.2 (ii) we get for $f \in B$ and $x_0 \in]0, 1[$ that

$$1 \geq \frac{1}{\pi} f(x_0) \int_0^1 \sin^2(\pi x) dx = \frac{1}{2\pi} f(x_0),$$

so the functions in B are uniformly bounded by 2π .

It is therefore clear that B is a compact convex base for W .

The extreme points of B are exactly the intersections between B and the extreme rays of W . We see that $2 \sin(\pi x) \in B$.

We claim that the following formulas hold, cf. [4]:

$$(3.1) \quad A_n^*(x) = \frac{2}{\pi^{2n+1}} \sum_{k=1}^{\infty} \frac{\sin(k\pi x)}{k^{2n+1}}, \quad n \geq 0, \quad x \in]0, 1[,$$

$$(3.2) \quad A_n(x) = \frac{2}{\pi^{2n+1}} \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\sin(k\pi x)}{k^{2n+1}}, \quad n \geq 0, \quad x \in]0, 1[.$$

Formula (3.2) follows immediately from (3.1). For $n = 0$ (3.1) is the familiar formula

$$\frac{\pi}{2} (1-x) = \sum_{k=1}^{\infty} \frac{\sin (k\pi x)}{k}, \quad 0 < x < 1.$$

Suppose that (3.1) holds for n replaced by $n - 1$ for some $n \geq 1$. Denoting the right-hand side of (3.1) by f_n , we have $f_n(0) = f_n(1) = 0$ and

$$f_n''(x) = -\frac{2}{\pi^{2n-1}} \sum_{k=1}^{\infty} \frac{\sin (k\pi x)}{k^{2n-1}}.$$

which is equal to $-\Lambda_{n-1}^*$ by the induction hypothesis. It follows by (2.3) that $f_n = \Lambda_n^*$, and (3.1) is proved. From (3.1) and (3.2) it follows that $\pi^{2n+1}\Lambda_n$ and $\pi^{2n+1}\Lambda_n^* \in B$. We also get $\lim_{n \rightarrow \infty} \pi^{2n+1}\Lambda_n(x) = \lim_{n \rightarrow \infty} \pi^{2n+1}\Lambda_n^*(x) = 2 \sin (\pi x)$. We have now established the following result:

PROPOSITION 3.1. *The set B is a compact convex base for W and the extreme points of B are $2 \sin (\pi x)$, $\pi^{2n+1}\Lambda_n^*(x)$, $\pi^{2n+1}\Lambda_n(x)$, $n \geq 0$, which form a closed subset of B .*

By l_+^1 we denote the set of sequences $(\alpha_n)_{n \geq 0}$ of non-negative numbers such that $\sum_0^{\infty} \alpha_n < \infty$.

By the Choquet representation theorem or just by the Krein-Milman theorem we get the following, cf. [3]:

THEOREM 3.2. *For every $f \in W$ there exist $a \geq 0$ and sequences $(\alpha_n), (\beta_n) \in l_+^1$ such that*

$$(3.1) \quad f(x) = 2a \sin (\pi x) + \sum_{n=0}^{\infty} \alpha_n \pi^{2n+1} \Lambda_n^*(x) + \sum_{n=0}^{\infty} \beta_n \pi^{2n+1} \Lambda_n(x); \quad 0 < x < 1.$$

The functions in B are uniformly bounded by 2π , and therefore the series (3.1) is uniformly convergent.

If we differentiate the series in (3.1) two times and change sign we get the series

$$\pi^2 \left(2a \sin (\pi x) + \sum_{n=0}^{\infty} \alpha_{n+1} \pi^{2n+1} \Lambda_n^*(x) + \sum_{n=0}^{\infty} \beta_{n+1} \pi^{2n+1} \Lambda_n(x) \right),$$

which also converges uniformly on $]0, 1[$ because $\sum_{n=0}^{\infty} \alpha_{n+1} + \sum_{n=0}^{\infty} \beta_{n+1} < \infty$.

It follows that the following formula holds:

$$(3.2) \quad (-1)^k f^{(2k)}(x) = \pi^{2k} \left(2a \sin(\pi x) + \sum_{n=0}^{\infty} \alpha_{n+k} \pi^{2n+1} A_n^*(x) + \sum_{n=0}^{\infty} \beta_{n+k} \pi^{2n+1} A_n(x) \right)$$

for $0 < x < 1$, $k \geq 0$ and furthermore

$$(3.3) \quad \alpha_k = \pi^{-2k-1} (-1)^k f^{(2k)}(0), \quad \beta_k = \pi^{-2k-1} (-1)^k f^{(2k)}(1) \\ \text{for } k \geq 0.$$

This proves that the sequences (α_n) , (β_n) and hence also a are uniquely determined by f . We have thus shown that B is a simplex. The extreme points of B form a closed subset of B as remarked in Proposition 3.1 so we can formulate the following

COROLLARY 3.3. *The base B for W is a Bauer simplex.*

Whittaker proved in [4] that the series in (3.1) in fact converges uniformly over arbitrary compact subsets of the complex plane. This also proves that f can be extended to an entire holomorphic function which we also call f . For $x \in]0, 1[$ and $y \in \mathbf{R}$ we then have

$$f(x + iy) = \sum_{k=0}^{\infty} f^{(k)}(x) \frac{(iy)^k}{k!},$$

hence

$$\operatorname{Re} f(x + iy) = \sum_{k=0}^{\infty} (-1)^k f^{(2k)}(x) \frac{y^{2k}}{(2k)!},$$

which shows that $x \mapsto \operatorname{Re} f(x + iy)$ belongs to W for all $y \in \mathbf{R}$, as sum of the functions

$$x \mapsto (-1)^k f^{(2k)}(x) \frac{y^{2k}}{(2k)!}$$

which all belong to the closed cone W .

This gives a short proof of the recent result of Mugler [2].

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