1. Completely convex functions

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1. Completely convex functions

Let I denote an open interval. A function $f: I \to \mathbf{R}$ is called *completely convex*, if it is C^∞ and $(-1)^k f^{(2k)} \geqq 0$ on I for $k \geqq 0$.

The set of completely convex functions is a convex cone denoted $W = W(I)$. We always equip W with the topology of pointwise convergence, i.e., with the topology induced by the product space $\mathbf{R}^I$.

Lemma 1.1. *If I is unbounded $W(I)$ consists of the non-negative affine functions, and $W(\mathbf{R})$ consists of the non-negative constants.*

Proof. Assume first that $\inf I = -\infty$. Then every $f \in W$ is decreasing since it is non-negative and concave. For $k \geqq 0$ and $f \in W$ we have $(-1)^k f^{(2k)} \in W$ and consequently $(-1)^k f^{(2k+1)} \leqq 0$. This shows that also $-f' \in W$ and then $-f'' \leqq 0$, but by definition $f'' \leqq 0$ and therefore f is affine.

The case $\sup I = \infty$ is treated in a similar manner. Finally, every non-negative concave function on $\mathbf{R}$ is constant.

Remark. Completely convex sequences are non-negative and affine.

For a sequence $a = (a_0, a_1, \ldots)$ of real numbers we define Δa to be the sequence $(\Delta a)_n = a_{n+1} - a_n$, $n \geqq 0$, and $\Delta^k a$ is defined as $\Delta(\Delta^{k-1} a)$ for $k \geqq 1$, where $\Delta^0 a = a$. A sequence a is called *completely convex* if $(-1)^k \Delta^{2k} a \geqq 0$ for $k \geqq 0$. The same method as in Lemma 1.1 leads to the conclusion that every completely convex sequence satisfies $\Delta a \geqq 0$ and $\Delta^2 a = 0$. The completely convex sequences are therefore exactly the sequences $a_n = \alpha n + \beta$, where $\alpha, \beta \geqq 0$.

This is an answer to a remark by Boas [1]: “Nothing seems to be known about completely convex sequences”.

In the following we will always assume that I is bounded, and for the sake of convenience we choose I to be $I =]0, 1[$. We simply write W for $W(]0, 1[)$. For $f \in W$ we have $-f'' \in W$ and $f^* \in W$, where f^* is defined by $f^*(x) = f(1-x)$. The mapping $f \mapsto f^*$ is an affine isomorphism of W onto itself.

Lemma 1.2. *Let $f:]0, 1[\to \mathbf{R}$ be non-negative and concave. Then the following holds:*

(i) $f(x) \leqq 2f(\tfrac{1}{2})$ *for* $x \in]0, 1[$.

(ii) $f(x) \geqq \dfrac{1}{\pi} f(x_0) \sin(\pi x)$ *for* $x, x_0 \in]0, 1[$.

(iii) ([6], Lemma 7.1) *If there exists* $x_0 \in]0, 1[$ *and* $a > 0$ *such that* $f(x_0) < a \sin(\pi x_0)$ *then* $f(x) \leqq a\pi$ *for* $x \in]0, 1[$.

Proof. (i). For $x \in]0, \frac{1}{2}]$ we have that $f(x)$ lies below the line through $(\frac{1}{2}, f(\frac{1}{2}))$ and $(1, 0)$ and (i) follows for $x \in]0, \frac{1}{2}]$. The interval $[\frac{1}{2}, 1[$ is treated similarly.

(ii). Let $x_0 \in]0, 1[$. For $x \in]0, x_0]$ we have

$$f(x) \geqq \frac{f(x_0)}{x_0} x \geqq f(x_0)\, x \geqq \frac{f(x_0)}{\pi} \sin(\pi x),$$

and for $x \in [x_0, 1[$ we have

$$f(x) \geqq \frac{f(x_0)(1-x)}{1-x_0} \geqq f(x_0)(1-x) \geqq \frac{f(x_0)}{\pi} \sin \pi(1-x) = \frac{f(x_0)}{\pi} \sin(\pi x).$$

(iii). If $f(x_0) > a\pi$ the inequality (ii) implies that $f(x) > a \sin(\pi x)$ for $x \in]0, 1[$.

Since every $f \in W$ can be extended to an entire holomorphic function all derivatives of f have finite limits at 0 and 1. This can also be established in an elementary way from the property $(-1)^k f^{(2k)} \geqq 0$ for $k \geqq 0$. We will therefore freely use $f^{(k)}(x)$ for $x = 0, 1$ as the limit of $f^{(k)}(x)$ at these points.

LEMMA 1.3. *The cone W is a closed and metrizable subset of* $\mathbf{R}^I$.

Proof. The set of concave functions $f: I \to \mathbf{R}$ is a closed and metrizable subset of $\mathbf{R}^I$, and therefore it suffices to prove that the pointwise limit f of a sequence (f_n) from W belongs to W.

It follows by Lemma 1.2 (i) that there exists a constant A such that $f_n \leqq A$ for all n [1]). The dominated convergence theorem then shows that

$$\lim_{n \to \infty} \int_0^1 f_n(x)\, \varphi(x)\, dx = \int_0^1 f(x)\, \varphi(x)\, dx$$

for all $\varphi \in \mathcal{D}(]0, 1[)$, so (f_n) converges to f weakly in the distribution sense. Therefore $(-1)^k f^{(2k)} \geqq 0$ for all $k \geqq 0$ in the distribution sense, and this implies that f is C^∞ and hence $f \in W$.

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[1]) In fact, $A = 2 \sup f_n(\frac{1}{2})$ can be used. It is finite because $\lim_{n \to \infty} f_n(\frac{1}{2})$ exists.