

# 1. Completely convex functions

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## 1. COMPLETELY CONVEX FUNCTIONS

Let  $I$  denote an open interval. A function  $f : I \rightarrow \mathbf{R}$  is called *completely convex*, if it is  $C^\infty$  and  $(-1)^k f^{(2k)} \geq 0$  on  $I$  for  $k \geq 0$ .

The set of completely convex functions is a convex cone denoted  $W = W(I)$ . We always equip  $W$  with the topology of pointwise convergence, i.e., with the topology induced by the product space  $\mathbf{R}^I$ .

**LEMMA 1.1.** *If  $I$  is unbounded  $W(I)$  consists of the non-negative affine functions, and  $W(\mathbf{R})$  consists of the non-negative constants.*

*Proof.* Assume first that  $\inf I = -\infty$ . Then every  $f \in W$  is decreasing since it is non-negative and concave. For  $k \geq 0$  and  $f \in W$  we have  $(-1)^k f^{(2k)} \in W$  and consequently  $(-1)^k f^{(2k+1)} \leq 0$ . This shows that also  $-f' \in W$  and then  $-f'' \leq 0$ , but by definition  $f'' \leq 0$  and therefore  $f$  is affine.

The case  $\sup I = \infty$  is treated in a similar manner. Finally, every non-negative concave function on  $\mathbf{R}$  is constant.

*Remark.* Completely convex sequences are non-negative and affine.

For a sequence  $a = (a_0, a_1, \dots)$  of real numbers we define  $\Delta a$  to be the sequence  $(\Delta a)_n = a_{n+1} - a_n, n \geq 0$ , and  $\Delta^k a$  is defined as  $\Delta(\Delta^{k-1} a)$  for  $k \geq 1$ , where  $\Delta^0 a = a$ . A sequence  $a$  is called *completely convex* if  $(-1)^k \Delta^{2k} a \geq 0$  for  $k \geq 0$ . The same method as in Lemma 1.1 leads to the conclusion that every completely convex sequence satisfies  $\Delta a \geq 0$  and  $\Delta^2 a = 0$ . The completely convex sequences are therefore exactly the sequences  $a_n = \alpha n + \beta$ , where  $\alpha, \beta \geq 0$ .

This is an answer to a remark by Boas [1]: “Nothing seems to be known about completely convex sequences”.

In the following we will always assume that  $I$  is bounded, and for the sake of convenience we choose  $I$  to be  $I = ]0, 1[$ . We simply write  $W$  for  $W(]0, 1[)$ . For  $f \in W$  we have  $-f'' \in W$  and  $f^* \in W$ , where  $f^*$  is defined by  $f^*(x) = f(1-x)$ . The mapping  $f \mapsto f^*$  is an affine isomorphism of  $W$  onto itself.

**LEMMA 1.2.** *Let  $f : ]0, 1[ \rightarrow \mathbf{R}$  be non-negative and concave. Then the following holds :*

$$(i) \quad f(x) \leq 2f(\tfrac{1}{2}) \quad \text{for } x \in ]0, 1[ .$$

$$(ii) \quad f(x) \geq \frac{1}{\pi} f(x_0) \sin (\pi x) \quad \text{for } x, x_0 \in ]0, 1[ .$$

(iii) ([6], Lemma 7.1) If there exists  $x_0 \in ]0, 1[$  and  $a > 0$  such that  $f(x_0) < a \sin(\pi x_0)$  then  $f(x) \leq a\pi$  for  $x \in ]0, 1[$ .

*Proof.* (i). For  $x \in ]0, \frac{1}{2}]$  we have that  $f(x)$  lies below the line through  $(\frac{1}{2}, f(\frac{1}{2}))$  and  $(1, 0)$  and (i) follows for  $x \in ]0, \frac{1}{2}]$ . The interval  $[\frac{1}{2}, 1[$  is treated similarly.

(ii). Let  $x_0 \in ]0, 1[$ . For  $x \in ]0, x_0]$  we have

$$f(x) \geq \frac{f(x_0)}{x_0} x \geq f(x_0) x \geq \frac{f(x_0)}{\pi} \sin(\pi x),$$

and for  $x \in [x_0, 1[$  we have

$$\begin{aligned} f(x) &\geq \frac{f(x_0)(1-x)}{1-x_0} \geq f(x_0)(1-x) \geq \frac{f(x_0)}{\pi} \sin \pi(1-x) = \\ &\quad \frac{f(x_0)}{\pi} \sin(\pi x). \end{aligned}$$

(iii). If  $f(x_0) > a\pi$  the inequality (ii) implies that  $f(x) > a \sin(\pi x)$  for  $x \in ]0, 1[$ .

Since every  $f \in W$  can be extended to an entire holomorphic function all derivatives of  $f$  have finite limits at 0 and 1. This can also be established in an elementary way from the property  $(-1)^k f^{(2k)} \geq 0$  for  $k \geq 0$ . We will therefore freely use  $f^{(k)}(x)$  for  $x = 0, 1$  as the limit of  $f^{(k)}(x)$  at these points.

LEMMA 1.3. *The cone  $W$  is a closed and metrizable subset of  $\mathbf{R}^I$ .*

*Proof.* The set of concave functions  $f : I \rightarrow \mathbf{R}$  is a closed and metrizable subset of  $\mathbf{R}^I$ , and therefore it suffices to prove that the pointwise limit  $f$  of a sequence  $(f_n)$  from  $W$  belongs to  $W$ .

It follows by Lemma 1.2 (i) that there exists a constant  $A$  such that  $f_n \leq A$  for all  $n$ <sup>1)</sup>. The dominated convergence theorem then shows that

$$\lim_{n \rightarrow \infty} \int_0^1 f_n(x) \varphi(x) dx = \int_0^1 f(x) \varphi(x) dx$$

for all  $\varphi \in \mathcal{D}(]0, 1[)$ , so  $(f_n)$  converges to  $f$  weakly in the distribution sense. Therefore  $(-1)^k f^{(2k)} \geq 0$  for all  $k \geq 0$  in the distribution sense, and this implies that  $f$  is  $C^\infty$  and hence  $f \in W$ .

<sup>1)</sup> In fact,  $A = 2 \sup f_n(\frac{1}{2})$  can be used. It is finite because  $\lim_{n \rightarrow \infty} f_n(\frac{1}{2})$  exists.