2. Determination of the extreme rays of W

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Inspired by [6] we consider the Green's function

$$G(x,t) = \begin{cases} (1-x)\,t & \text{for} \quad 0 \leqq t < x \leqq 1\,, \\ (1-t)\,x & \text{for} \quad 0 \leqq x \leqq t \leqq 1\,. \end{cases}$$

If φ is a continuous function on [0, 1] the unique solution $f \in C([0,1]) \cap C^2(]0,1[)$ to the equations

$$(2.1) \qquad f'' = -\varphi \text{ in }]0,1[\,, \quad f(0) = f(1) = 0$$

is

$$(2.2) \qquad f(x) = \int_0^1 G(x,t)\,\varphi(t)\,dt\,.$$

The successive iterates of G are defined for $x, t \in [0,1]$ by the equations

$$G_1(x,t) = G(x,t)$$

$$G_n(x,t) = \int_0^1 G(x,y)\,G_{n-1}(y,t)]\,dy\,, \quad n \geqq 2\,.$$

It is clear that $G_n(x,t) \geqq 0$ for $x, t \in [0,1]$.

We define recursively a sequence of polynomials $(\Lambda_n)_{n \geqq 0}$ [1]) by the requirement

$$(2.3) \qquad \Lambda_0(x) = x,\ \Lambda_n'' = -\Lambda_{n-1} \quad \text{and} \quad \Lambda_n(0) = \Lambda_n(1) = 0 \quad \text{for} \quad n \geqq 1\,.$$

The polynomial Λ_n is of degree $(2n+1)$, and we clearly have

$$(2.4) \qquad \Lambda_n(x) = \int_0^1 G(x,t)\,\Lambda_{n-1}(t)\,dt = \int_0^1 G_n(x,t)\,t\,dt \quad \text{for} \quad n \geqq 1\,,\ x \in [0,1]\,.$$

It follows that $\Lambda_n \geqq 0$ on [0, 1] for all n, and since $(-1)^k \Lambda_n^{(2k)} = \Lambda_{n-k}$ for $k \leqq n$ we see that $\Lambda_n \in W$.

We recall that a ray $\mathbf{R}_+ x$ of a cone C is called *extreme*, if an equation $x = f + g$ with $f, g \in C$ is possible only if $f, g \in \mathbf{R}_+ x$, cf. [3].

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[1]) Our terminology is different from that of [6]; $(-1)^n \Lambda_n$ is equal to the n'th Lidstone polynomial of [4] and [6].

PROPOSITION 2.1. *The polynomials Λ_n, $n \geqq 0$, lie on extreme rays of W.*

Proof. If $\Lambda_0 = f + g$ with $f, g \in W$ we have $0 = f'' + g''$, but since f'' and g'' are both $\leqq 0$, we conclude that f and g are affine. Furthermore, since $f(0) = g(0) = 0$, we conclude that f and g are proportional to Λ_0.

Suppose now that Λ_{n-1}, $n \geqq 1$, lies on an extreme ray of W, and assume that $\Lambda_n = f + g$ where $f, g \in W$. Then $\Lambda_{n-1} = -f'' + (-g'')$, and the induction hypothesis implies that $-f''$ and $-g''$ are proportional to Λ_{n-1}. Therefore we have $f = \lambda \Lambda_n(x) + ax + b$ for certain numbers $\lambda \geqq 0$, a, b. Since $0 \leqq f \leqq \Lambda_n$, we have $f(0) = f(1) = 0$ which implies that $a = b = 0$. This proves that f (and similarly g) are proportional to Λ_n which then lies on an extreme ray of W.

Since $f \mapsto f^*$ is an affine isomorphism of W the polynomials Λ_n^* also lie on extreme rays of W. The following result is a special case of [6], Theorem 1.1.

PROPOSITION 2.2. *Every function $f \in W$ can for $n \geqq 1$ be written as*

$$f(x) = \sum_{k=0}^{n-1} \left((-1)^k f^{(2k)}(0)\, \Lambda_k^*(x) + (-1)^k f^{(2k)}(1)\, \Lambda_k(x)\right) + R_n(x)\,,$$

where

$$R_n(x) = \int_0^1 G_n(x,t)\,(-1)^n f^{(2n)}(t)\,dt \in W\,.$$

Proof. For $n = 1$ the formula is equivalent with

$$(2.5) \quad f(x) - f(0)(1-x) - f(1)\,x = R_1(x) = -\int_0^1 G(x,t) f''(t)\,dt\,,$$

which follows directly from (2.2), and it is clear that $R_1 \in W$.

Suppose now the formula holds for some $n \geqq 1$. Applying (2.5) to $(-1)^n f^{(2n)} \in W$ we get

$$(-1)^n f^{(2n)}(x) = (-1)^n f^{(2n)}(0)\, \Lambda_0^*(x) + (-1)^n f^{(2n)}(1)\, \Lambda_0(x)$$
$$+ \int_0^1 G(x,t)\,(-1)^{n+1} f^{(2n+2)}(t)\,dt\,,$$

which substituted in the expression for R_n yields the formula for $n+1$ because of (2.4).

To see that $R_n \in W$ we notice that

$$(-1)^k R_n^{(2k)}(x) = \begin{cases} \displaystyle\int_0^1 G_{n-k}(x,t)\,(-1)^n f^{(2n)}(t)\,dt & \text{for} \quad 0 \leqq k \leqq n-1 \\ (-1)^k f^{(2k)}(x) & \text{for} \quad k \geqq n\,. \end{cases}$$

The following lemma is easy to establish, but instead of giving the proof here we refer to [6].

LEMMA 2.3. *There exists a constant* $M > 0$ *such that*

$$0 \leqq \int_0^1 G_n(x,t)\,dt \leqq \frac{M}{\pi^{2n}} \qquad \text{for} \qquad 0 \leqq x \leqq 1\,, \; n \geqq 1\,.$$

PROPOSITION 2.4. *The only functions* $f \in W$ *satisfying* $f^{(2k)}(0) = f^{(2k)}(1) = 0$ *for all* $k \geqq 0$ *are* $f(x) = a \sin(\pi x)$ *with* $a \geqq 0$.

Proof. Suppose $f \in W$ satisfies $f^{(2k)}(0) = f^{(2k)}(1) = 0$ for all $k \geqq 0$. Defining $a = \sup\{\alpha \geqq 0 \mid f - \alpha \sin(\pi x) \in W\}$, $g = f - a \sin(\pi x)$ belongs to W because W is closed in $\mathbf{R}^I$. Furthermore

$$g^{(2k)}(0) = g^{(2k)}(1) = 0 \text{ for all } k \geqq 0.$$

Let $\varepsilon > 0$ be given. Since $\varphi = g - \varepsilon \sin(\pi x) \notin W$, there exist $k \geqq 0$ and $x_0 \in]0, 1[$ such that $(-1)^k \varphi^{(2k)}(x_0) < 0$, hence

$$(-1)^k g^{(2k)}(x_0) < \varepsilon \pi^{2k} \sin(\pi x_0)\,.$$

By Lemma 1.2 (iii) applied to $(-1)^k g^{(2k)}$ we get

$$(-1)^k g^{(2k)}(t) \leqq \varepsilon \pi^{2k+1} \qquad \text{for} \qquad 0 < t < 1\,,$$

and therefore by Proposition 2.2 and Lemma 2.3 for $0 < x < 1$

$$g(x) = \int_0^1 G_k(x,t)(-1)^k g^{(2k)}(t)\,dt \leqq \varepsilon \pi^{2k+1} \int_0^1 G_k(x,t)\,dt$$
$$\leqq \varepsilon M \pi\,.$$

This proves that g is identically zero.

PROPOSITION 2.5. *The extreme rays of* W *are precisely the rays generated by* Λ_n *and* $\Lambda_{n'}^*$, *where* $n \geqq 0$, *and* $\sin(\pi x)$.

Proof. We first show that $\sin(\pi x)$ lies on an extreme ray. If $\sin(\pi x) = f + g$ where $f, g \in W$, we have $f(0) = f(1) = g(0) = g(1) = 0$. Differentiating $2k$ times we similarly get $f^{(2k)}(0) = f^{(2k)}(1) = g^{(2k)}(0) = g^{(2k)}(1) = 0$, and it follows by Proposition 2.4 that f and g are proportional to $\sin(\pi x)$.

We finally have to show that an arbitrary extreme ray is generated by one of the above functions.

Assume that $f \in W$ generates an extreme ray. If $f^{(2k)}(0) = f^{(2k)}(1) = 0$ for all $k \geqq 0$ we already know by Proposition 2.4 that f is proportional to $\sin(\pi x)$. Otherwise let n be the smallest number $\geqq 0$ for which $f^{(2n)}(0) \neq 0$ or $f^{(2n)}(1) \neq 0$. By Proposition 2.2 we then have

$$f(x) = (-1)^n f^{(2n)}(0)\, \Lambda_n^*(x) + (-1)^n f^{(2n)}(1)\, \Lambda_n(x) + R_{n+1}(x),$$

but since f lies on an extreme ray all three terms on the right-hand side lie on this ray.

If $f^{(2n)}(0) \neq 0$ this shows that $(-1)^n f^{(2n)}(1) \Lambda_n$ and R_{n+1} are proportional to Λ_n^*. Therefore $f^{(2n)}(1) = 0$ and $R_{n+1}^{(2n+2)} = f^{(2n+2)}$ is proportional to $(\Lambda_n^*)^{(2n+2)} = 0$, so that $f^{(2n+2)} = 0$ and hence $R_{n+1} = 0$ (cf. Proposition 2.2).

If $f^{(2n)}(1) \neq 0$ we similarly get $f^{(2n)}(0) = 0$ and $R_{n+1} = 0$. This shows that f lies on the ray generated by either Λ_n^* or Λ_n.

3. Determination of a base for W

There are several ways of determining a base for W. We choose the following set

$$B = \left\{ f \in W \mid \int_0^1 f(x) \sin(\pi x)\, dx = 1 \right\}.$$

By Lemma 1.2 (ii) we get for $f \in B$ and $x_0 \in]0, 1[$ that

$$1 \geqq \frac{1}{\pi} f(x_0) \int_0^1 \sin^2(\pi x)\, dx = \frac{1}{2\pi} f(x_0),$$

so the functions in B are uniformly bounded by 2π.

It is therefore clear that B is a compact convex base for W.

The extreme points of B are exactly the intersections between B and the extreme rays of W. We see that $2 \sin(\pi x) \in B$.

We claim that the following formulas hold, cf. [4]:

$$(3.1) \quad \Lambda_n^*(x) = \frac{2}{\pi^{2n+1}} \sum_{k=1}^{\infty} \frac{\sin(k\pi x)}{k^{2n+1}}, \quad n \geqq 0, \; x \in]0, 1[,$$

$$(3.2) \quad \Lambda_n(x) = \frac{2}{\pi^{2n+1}} \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\sin(k\pi x)}{k^{2n+1}}, \quad n \geqq 0, \; x \in]0, 1[.$$