1. Continuous solutions of Systems of linear equations

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **23 (1977)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **26.05.2024**

published 37 papers of which 24 were joint papers with myself (and sometimes with a third collaborator, too, principally Urs Stammbach). This long and fruitful collaboration is of course my own adequate testimony to the high regard in which I have always held my good friend Beno Eckmann. But it would perhaps contradict certain canons of good taste if I were to cite our joint work in evidence of the depth of penetration of Eckmann's mathematical insights. Let me therefore only say of that work that I regard my collaboration with Eckmann, and my previous apprenticeship as a student of Henry Whitehead, as the two principal formative elements in my own mathematical growth and maturity. I would only wish to add a reference to the gratification which Eckmann and I felt that a *leit motif* of our joint research, the heuristic duality which we uncovered at the heart of homotopy theory and exploited, received recognition from Norman Steenrod in his listing of principal themes of algebraic topology.

I have said that Eckmann remains as active in mathematical research as ever. This is a source of great delight to us gathered here for this congress, as also for the many mathematicians, all over the world, who derive benefit from his contributions to the progress of our subject. For it is not enough to say that Eckmann remains active; he remains effective, discriminating and entirely contemporary. His most recent work, with Robert Bieri, on Poincaré duality groups and a certain natural generalization of such groups, of which you may hear from Bieri at this congress, exhibits all the qualities to which I have already referred. It is a remarkable tribute to Beno Eckmann that one may say of him that today, at the age of 60, he is still doing his best work.

But, as I feel sure Beno would himself agree, we have had enough of generalities—it is time to get down to some mathematics!

1. Continuous solutions of systems of linear equations

In [8; 1943] Eckmann considered the following problem. Suppose given a system of r linear equations in n unknowns, $r < n$,

$$\sum_{k=1}^{n} a_{ik} x_k = 0, \quad i = 1, 2, \ldots, r < n, \tag{1.1}$$

where the coefficients a_{ik} are continuous real-valued [1]) functions of a variable u which describes some topological space U, which will usually

[1]) Eckmann had considered the corresponding problem in the complex case in [3; 1942].

be supposed compact metric. A *continuous* solution of the system (1.1) consists of n continuous real-valued functions $x_k(u)$ such that

$$\sum_{k=1}^{n} a_{ik}(u)\, x_k(u) = 0\,;$$

a system of solutions is *linearly independent* if, for each $u \in U$, the solutions are linearly independent in the usual sense; thus a single solution is linearly independent if it vanishes for no value of u. Eckmann supposes the coefficient matrix $(a_{ik}(u))$ to have maximum rank r for all $u \in U$ and asks how many linearly independent solutions the system admits. In answer, he first translates the problem into matrix language: Let $A_{n,r}$ map U to the space of $r \times n$ orthogonal matrices.[1]) Can $A_{n,r}$ be extended, by adjoining rows, to $A_{n,r+l}$? Clearly, if so, and if $A_{n,r}$ is obtained from the coefficient matrix of (1.1) by orthogonalization, then (1.1) admits l linearly independent solutions.

It is, however, Eckmann's next step which we should emphasize. If $m = r + l$, then we consider the fibre-map of Stiefel manifolds $V_{n,m} \xrightarrow{p} V_{n,r}$ with fibre $V_{n-r,m-r}$. Of course, the manifolds $V_{n,m}$ were not called Stiefel manifolds in those days, but Eckmann naturally referred to Stiefel's 1935 paper. Moreover, Eckmann spoke of the *factorization* or *decomposition* ("Zerlegung"),

$$\text{(1.2)} \qquad Z\colon\ V_{n,m} / V_{n-r,m-r} = V_{n,r}\,.$$

If $f\colon X \to V_{n,m}$ is a map, then f is called the *trace* of $F = Pf\colon X \to V_{n,r}$, with respect to Z, and the problem is to decide whether a given map $A_{n,r}\colon U \to V_{n,r}$ is a trace with respect to Z. Plainly this is a homotopy question, depending only on the homotopy class of $A_{n,r}$; plainly too a constant map is a trace. It follows that if U is contractible then every system (1.1) of maximum rank admits $n - r$ linearly independent solutions—this is a theorem of Wazewski.

Let us now take $U = S^q$. The problem is now one involving *homotopy groups*, and Eckmann comes extraordinarily close to writing down the homotopy sequence of the fibration (1.2); certainly he exploits it effectively, If $q = n - 1$, then we must study homotopy classes of maps $A_{n,m}\colon S^{n-1} \to V_{n,m}$. By projection we get an element of $\pi_{n-1}(V_{n,1})$, that is, an integer c, which we call the *characteristic* of $A_{n,m}$, and it is immediate that a matrix map $A_{n,m}$ exists, with $c = 1$, if and only if there is an $(m-1)$-field on S^{n-1}. Thus if an $(m-1)$-field exists on S^{n-1} every $A_{n,m}$ occurs so that (1.1) has $m - r$ linearly independent solutions.

---

[1]) We might expect the notation $A_{r,n}$; I have conserved Eckmann's notation.

One may show dy direct matrix arguments that, if $m \geqslant 2$, then $c = 0$ if n is odd (corresponding to the absence of a non-singular vector field on S^{n-1}) and that all even values of c occur if n is even. The question whether odd values of c occur reduces to the question whether $c = 1$ occurs and this in turn leads to the consideration of the Hurwitz-Radon Theorem (see Section 2). Eckmann uses the then existing, scanty knowledge of homotopy groups of Stiefel manifolds to obtain special results (when $q \neq n - 1$)—we would do the same today, but would benefit from our more extensive knowledge.

Indeed, Eckmann himself returned to the question 24 years later when he lectured at a Battelle Rencontre in Seattle [67; 1968]. By this time, of course, Adams had proved his celebrated *Hopf Invariant One Theorem* and the properties of topological K-theory had been substantially developed. Eckmann performed the significant feat of explaining the theory, and its applications—to systems of linear equations, to the existence of (generalized) vector products in $\mathbf{R}^n$, to the parallelizability of spheres, and to the existence of almost-complex structures on spheres—of explaining all this to an audience dominated by theoretical physicists! What testimony to his clarity—and courage!

2. A group theoretical proof of the Hurwitz-Radon Theorem

Immediately following the work discussed above, Eckmann produced [9; 1943] a truly beautiful proof of the celebrated theorem on the composition of quadratic forms. The problem is to determine, given n, those values of p such that there exist n bilinear forms $z_1, \ldots, z_n$ of the variables $x_1, \ldots, x_p$; $y_1, \ldots, y_n$, with complex coefficients, such that the identity

$$(2.1) \qquad (x_1^2 + \ldots + x_p^2)\,(y_1^2 + \ldots + y_n^2) = z_1^2 + \ldots + z_n^2$$

holds. As formulated by Radon in 1923, the solution is the following. Let $n = u \,.\, 2^{4\alpha+\beta}$ with u odd and $0 \leqslant \beta \leqslant 3$. Then we can find $z_1, \ldots, z_n$ to satisfy (2.1) if and only if $p \leqslant 8\alpha + 2^\beta$. Actually, Radon considered forms with real coefficients, but Eckmann showed explicitly in his proof that a solution of (2.1) for forms with complex coefficients implies a solution for forms with real coefficients. Eckmann's proof is based on the classical theory of (complex) representations of finite groups, together with certain particular results, due to Frobenius and Schur, relating complex to real representations. Before outlining Eckmann's proof, let me quote Eck-