3. Complexes with operators

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represented by I. Thus the degree n of an arbitrary representation of G of the required kind is given by

$$n = m \,.\, 2^{\frac{p-1}{2}},\ p \text{ odd}; \quad n = m \,.\, 2^{\frac{p-2}{2}},\ p \text{ even}. \tag{2.4}$$

It remains to determine which of those representations are equivalent to an orthogonal representation—these will also, according to Frobenius-Schur, be equivalent to orthogonal *real* representations. Corresponding to an irreducible representation D of G, one computes $S = \sum_{g \in G} \chi(g^2)$, where χ is the character function. Then D is real-equivalent if and only if $S > 0$; D is equivalent to its complex conjugate $\bar{D}$ if $S < 0$; and D is not equivalent to $\bar{D}$ if $S = 0$. By a very beautiful application of the elementary theory of complex numbers, Eckmann used this criterion to show that the given irreducible representations of G (whereby ε is represented by $-I$) are real-equivalent (that is, orthogonal-equivalent) if $p \equiv 7, 0, 1 \bmod 8$, and not otherwise. If $p \equiv 3, 4, 5 \bmod 8$ they are equivalent to their complex conjugates; if $p \equiv 2, 6 \bmod 8$ they are not. One may immediately deduce the degrees of real-irreducible real representations of G, and hence show that for a given $n = u \,.\, 2^t$, with u odd, the maximum value of p such that there exists a real (orthogonal) representation of G of degree n, in which ε is represented by $-I$, is given by the rule:

$$\begin{aligned} t &= 4\alpha &&: p = 8\alpha + 1 \\ t &= 4\alpha + 1 &&: p = 8\alpha + 2 \\ t &= 4\alpha + 2 &&: p = 8\alpha + 4 \\ t &= 4\alpha + 3 &&: p = 8\alpha + 8\,. \end{aligned}$$

This is the Hurwitz-Radon Theorem. Today we know that, when translated into the language of vector fields on spheres, the Hurwitz-Radon number $p - 1$ provides an upper bound on the number of vector fields on S^{n-1} even without the linearity condition; this was, of course, proved by Adams exploiting the techniques of topological K-theory.

3. Complexes with operators

Here perhaps I trespass somewhat on Saunders MacLane's territory. But I do want to exemplify a characteristic feature of Eckmann's thought, whereby he passes freely to and fro between topology and algebra, generalizing both aspects in a constructive and purposeful way. In [33; 1953], which was really the sequel to a pair of short papers [17, 18; 1947] which had appeared some years earlier in the Proceedings of the National Academy of Sciences, Eckmann considered a generalization of the algebraic constructions involved in studying the homology of covering spaces.

Let R be a unitary ring and let C be an R-complex (that is, a chain complex such that each C_p is an R-module and each boundary $\partial: C_p \to C_{p-1}$ an R-homomorphism). Let J be an abelian group and let $\Phi = \operatorname{Hom}(R, J)$ be the group of additive homomorphisms of R into J, turned into an R-module by the rule

$$(s\varphi)(r) = \varphi(rs), r, s \in R, \varphi: R \to J \tag{3.1}$$

If Ψ is a submodule of Φ we say that the p-cochain f of C, with values in J, is of type Ψ if, for each $c_p \in C_p, f(rc_p)$, as a function of $r \in R$, belongs to Ψ. It is easy to check that then the coboundary δf is again of type Ψ, so that we may define the Ψ-cohomology groups of C with coefficients in J, written $H^p_\Psi(C, J)$. Among the examples which Eckmann gives of Ψ-cohomology are the following:

(a) If $\Psi = \Phi$, we simply get the cohomology groups $\check{H}^p(C, J)$ of C, regarded as a complex of abelian groups, with values in J.

(b) If J is an R-module and Ψ consists of the R-homomorphisms from R to J, then a cochain of type Ψ is an *equivariant* cochain and $H^p_\Psi(C, J)$ is just the equivariant cohomology group which we will write simply as $H^p(C, J)$. Clearly we have here an isomorphism $\Psi \cong J$ given by $\varphi \mapsto \varphi(1)$. This isomorphism suggests the general conclusion embodied in the isomorphism (3.2) below.

(c) If Q is a subring of R and J is a Q-module, we may take Ψ to consist of all Q-homomorphisms $R \to J$. If C_Q denotes the complex C regarded as a Q-complex, then $H^p_\Psi(C, J) = H^p(C_Q, J)$. Plainly, (c) generalizes (a) and (b).

(d′) Let A be a group, B be a subgroup of A. If we take $R = \mathbf{Z}[A]$, $Q = \mathbf{Z}[B]$ in (c) we obtain the group Ψ of functions ψ from A to the B-module J such that $\psi(ba) = b\psi(a)$. The A-module structure on Ψ is said to be *induced* by the B-module structure on J (it corresponds very precisely to the *induced representation* of A, induced by the representation of B by J).

(d″) If A is a group and J an abelian group, and if $R = \mathbf{Z}[A]$ we may take Ψ to consist of those functions $\Psi: A \to J$ which vanish almost every-where on A. The resulting cohomology group $H^p_\Psi(C, J)$ is called *A-finite* and denoted $H^p_{A-\mathrm{fin}}(C, J)$. If $J = \mathbf{Z}$, $\Psi \cong \mathbf{Z}[A]$; an isomorphism (of A-modules) is given by $\psi \to \Sigma\ \psi(a)\, a^{-1}$.

Eckmann unifies all those examples, coalescing them into example (b), by means of the isomorphism

$$(3.2) \qquad H^p(C, \Psi) \cong H^p_\Psi(C, J),$$

induced, at the cochain level, by $f \mapsto g$, where

$$(3.3) \quad g(c) = f(c)(1),\ c \in C_p,\ f\colon C_p \to \Psi \text{ (equivariant)},\ g\colon C_p \to J;$$

he then applies (3.2) in various contexts. The fact that the isomorphism (3.2) is now a commonplace certainly does not detract from its significance—on the contrary!

Among the applications, let us mention the following. Les S be a nice topological space, so that we can construct covering spaces of S. Let B be a subgroup of the fundamental group A of S, let S_B be the covering space of S corresponding to B, let J be a B-module and let Ψ be the induced A-module in the sense of (d'). We then have an isomorphism of singular cohomology with local coefficients,

$$(3.4) \qquad H^p(S_B, J) \cong H^p(S, \Psi).$$

A further, very significant application made by Eckmann in [33] and further developed in [35; 1953] is to the (generalized) *transfer*. We will not go into that here, but instead will turn to the theory of *ends* of groups. If A is a finitely presented group and P a compact polyhedron with $\pi_1 P = A$, then the ends of A may, following Hopf, be defined in terms of the universal cover $\tilde{P}$ of P; since they refer to the "infinite components" of $\tilde{P}$, the theory of ends is only of interest if A is not finite. If C is the chain-complex of some simplicial decomposition of $\tilde{P}$, then Specker proved that $H^1(C, \mathbf{Z}[A])$ is a free abelian group whose rank is $e - 1$, where e is the number of ends [1]) of A. Indeed, one has

$$(3.5) \qquad H^1(C, \mathbf{Z}[A]) \cong H^1(A, \mathbf{Z}[A]) \cong D/D_0,$$

where D is the group of (Fox) *derivations* from A to $\mathbf{Z}[A]$ and D_0 is the subgroup of inner derivations. Now according to (d'') and (3.2), $H^1(C, \mathbf{Z}[A])$

[1]) Hopf proved that, if A is not finite, then e can only take the values $1, 2, \infty$.

$\cong H^1_{A-\text{fin}}(C, \mathbf{Z})$. But since the chain complex C/A, obtained from C by factoring out the operations of A, is of finite type, it follows that

$$H^1_{A-\text{fin}}(C, \mathbf{Z}) = H^1_{\text{fin}}(C, \mathbf{Z}), \tag{3.6}$$

the first cohomology group of $\tilde{P}$ with integer coefficients *based on finite cochains*, that is, on cochains which vanish on almost every simplex of $\tilde{P}$.

4. SPACES WITH MEANS

In his talk at the 1950 International Congress of Mathematicians [32], the first international congress to be held after the second world war, Eckmann addressed himself to the question of the existence on a topological space X of a map $\mu \colon X^n \to X$, where X^n is the n-th cartesian power of X, which should be symmetric in the n variables and should satisfy $\mu(x, x, \ldots, x) = x, x \in X$. He returned to the theme in the paper he presented on the occasion of the celebration of the sixtieth birthday of Heinz Hopf [39; 1954], and it is therefore appropriate that I should refer to it here.

The methods used by Eckmann to study this problem were, of course, those of homotopy theory; they are thus very different from those of Aumann who first considered the problem in 1943. On the other hand, they do enable one to investigate the more general concept of *homotopy-mean*, whereby we understand that the map μ is only required to satisfy the conditions imposed above *up to homotopy*. This approach was explicitly followed in the sequel [55; 1962] where, in collaboration with T. Ganea and the present writer, Eckmann effectively gave a complete solution of the problem, or, as one may say, killed it!

In [39], Eckmann showed that if X admits an n-mean, so do its homotopy groups and homology groups. So far as the homotopy groups are concerned this is a "trivial" consequence, in the sense that it follows on categorical grounds from the fact that the homotopy group functor is *product-preserving*; however, the argument relating to the (integer-valued) homology groups was a rather subtle application of the Künneth Theorem. Moreover, an n-mean, $n \geqslant 2$, can only exist in a group if the group is abelian, and then it exists (and is unique) if and only if the group admits unique division by n. Indeed the n-mean $\mu \colon G^n \to G$ is simply

$$\mu(x_1, x_2, \ldots, x_n) = \frac{1}{n}(x_1 + x_2 + \ldots + x_n). \tag{4.1}$$

Eckmann used this criterion in establishing that the homology groups of X admitted an n-mean if X admitted an n-mean; and to show that if the compact polyhedron X admits a (homotopy) n-mean for all n, then X is contractible. He raised many questions, among them whether the existence in such a space X of an n-mean for some $n \geqslant 2$ might imply the contractibility of X. This question was answered positively in [55].

In that paper, the idea of an n-mean was first placed in its appropriate categorical setting, so that the trivial (= categorical) aspects of the theory of n-means could first be exhibited. In particular this permitted the description of the dual concept of an *n-comean*. However, insofar as groups are concerned, the situation for the existence of n-comeans must be distinguished from that for the existence of n-means. Fix $n \geqslant 2$. Then in the category of groups, only the trivial group admits an n-comean; in the category of abelian groups, A admits an n-comean if and only if it admits an n-mean. On the other hand, there are many compact polyhedra admitting an n-comean; if m is prime to n and $k \geqslant 2$, and if X is the Moore space having $\mathbf{Z}/m$ as its single non-vanishing homology group in dimension k, then X admits an n-comean. On the other hand, the dual of Eckmann's result in [39] holds: if the compact polyhedron X admits an n-comean for all n, then X is contractible.

Reverting to Eckmann's question, one shows first that if X is a compact polyhedron admitting an n-mean, $n \geqslant 2$, then X is an H-space; that is, X admits a continuous multiplication with two-sided unity element. However, Browder showed that such spaces satisfy Poincaré duality. Thus, were X non-contractible, its top-dimensional (non-vanishing) homology group $H_k X$ would be cyclic infinite, and therefore could not admit division by n, contradicting Eckmann's result in [39]. Thus the application of Browder's deep theorem kills the homotopy-theoretic interest of means in compact polyhedra. It is pleasant to record that the obituary article [55] was published in *Studies in Mathematical Analysis and Related Topics*, symbolic testimony to the value of Eckmann's constant search for relations between the different branches of mathematics. But perhaps, like a famous obituary of Bertrand Russell, the notice of the death of homotopy n-means was premature. For localization theory, a powerful new tool in homotopy theory, has rekindled interest in non-compact polyhedra, and there are surely interesting examples of such polyhedra admitting n-means—for example, Eilenberg-MacLane spaces for $\mathbf{Z}\left[\frac{1}{n}\right]$-local spaces.