1. Introduction

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ALTERNATIVE HOMOTOPY THEORIES [1]

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Dedicated to Beno Eckmann on the occasion of his 60th birthday

1. Introduction

Recently there has been considerable interest in the theory of G-spaces, where G is a topological group. The purpose of this lecture is to describe some of the work that has been done at Oxford in the past few years, particularly work concerned with equivariant homotopy theory and the associated homotopy theory of spaces over a given space. Little is known about these alternative homotopy theories outside the "stable range". Special emphasis will therefore be placed on non-stable questions, such as the existence of Hopf structures and the Whitehead product theory. Before embarking on this, however, I would like to make a few preliminary remarks.

Let us begin by considering the category of (right) G-spaces, where G is a topological group. Both the product $\times$ and the join $*$ are defined in this category. Among the concepts which seem to belong here is that of group G-space. We say that a G-space A is a *group G-space* if G is a topological group with equivariant multiplication $A \times A \to A$. This implies, of course, that inversion is also equivariant and that the neutral element e is a fixed point. Note that G itself constitutes a group G-space under the action of conjugation.

Let $f : X \to Y$ be a G-map, where X and Y are G-spaces. Let $f^H : X^H \to Y^H$ denote the corresponding map of the fixed-point sets, for any subgroup $H \subset G$. Clearly f^H is a homotopy equivalence if f is a G-homotopy equivalence. Recently Segal [13] has proved that conversely f is a G-homotopy equivalence provided (i) X and Y are G-ANR's, (ii) G is a compact Lie group and (iii) f^H is a homotopy equivalence for every closed subgroup H of G. This important theorem enables many results of ordinary homotopy theory to be generalized to equivariant homotopy theory.

[1]) Presented at the Colloquium on Topology and Algebra, April 1977, Zurich.

In general it is difficult to say very much about the space of equivariant maps. However, the following example, due to George Wilson [19], is instructive [1]). Take $G = SO(q)$, the rotation group, and $A = S^{q-1}$, where $q \geqslant 1$. Let B and C be trivial G-spaces. Choose a point $a_0 \in S^{q-1}$ and consider both the cone $TB = \{a_0\} * B \subset S^{q-1} * B$ and the suspension $SC = \{\pm a_0\} * C \subset S^{q-1} * C$. Every G-map $f : S^{q-1} * B \to S^{q-1} * C$ is determined by its values on TB. Moreover $f \mid TB$ determines a map f': $(TB, B) \to (SC, C)$, from consideration of the fixed point sets of the isotropy subgroups, and conversely every such map f' determines a G-map f by $f(xg) = (f'x)\, g \; (x \in TB, g \in G)$.

Let P be a principal G-bundle over a space X. To every G-space A there is associated a G-bundle $P_{\#}A$ with fibre A, and similarly with G-maps. We refer to $P_{\#}$ as the *principal functor*. Note that trivial G-spaces transform into trivial bundles; thus the fixed point set A^G of A transforms into the trivial subbundle $P_{\#}A^G$ of $P_{\#}A$. In particular every fixed point $a \in A$ determines a cross-section $P_{\#}a$ of $P_{\#}A$.

Any self-functor F on the category of G-spaces can be extended to the category of G-bundles by defining $FP_{\#}A = P_{\#}FA$. Binary functors are treated similarly. Thus the product $\times$ and the join $*$ in the category of G-spaces transform, under $P_{\#}$, into the (fibre) product $\times$ and the (fibre) join $*$ in the category of G-bundles.

Here is a less familiar example, currently being investigated by my research student Duncan Harvey. Let A be a G-space with distinct fixed points $(a_1, \ldots, a_m)$. For $n = 1, 2, \ldots$ the configuration space $F_{n,m}A$ of A is defined as the space of n-tuples $(x_1, \ldots, x_n)$ of distinct points in $A - \{a_1, \ldots, a_m\}$. Regard $F_{n,m}A$ as a G-space with action as in A^n. Write $E_{n,m} = P_{\#}F_{n,m}A$, where $E = P_{\#}A$. Then $E_{n,m}$ can be described as the bundle whose fibre over the point $x \in X$ is the configuration space $F_{n,m}E_x$ relative to $(s_1 x, \ldots, s_m x)$, where $s_1 = P_{\#}a_1, \ldots, s_m = P_{\#}a_m$.

It may happen that a G-map is homotopic to the identity but not G-homotopic. In that case it is interesting to try and determine the principal G-bundles P with the property that the image of the G-map under $P_{\#}$ is fibre homotopic to the identity. This question has been studied in [8] for the antipodal map on the $SO(q)$-space S^{q-1} (q even), and we continue this investigation here from a rather different point of view.

In ordinary homotopy theory the advantages of introducing basepoints are well understood. In equivariant homotopy theory the advantages are

[1]) Actually [19] treats the case $q = 3$ when B and C are spheres.

similar. We work with the category of pointed G-spaces, i.e. G-spaces with fixed point. The suspension functor Σ and the loop-space functor Ω are then defined, also the binary functors wedge $\vee$ and smash $\wedge$. This corresponds, of course, to the category of sectioned G-bundles, i.e. G-bundles with cross-section, where these functors are also defined and commute with the principal functor. We prefer, however, to enlarge this to the category of ex-spaces — see § 4.

2. Equivariant homotopy theory

Let G be a topological group and let A_i $(i=1, 2)$ be a pointed G-space. The space of pointed G-maps $f: A_1 \to A_2$ is denoted by $M_G(A_1, A_2)$, and the set of pointed G-homotopy classes of pointed G-maps by $\pi_G(A_1, A_2)$. The class of the constant map $e: A_1 \to A_2$ is denoted by 0. In this context we reserve the symbol $\simeq$ for the relation of pointed G-homotopy.

Let A be a pointed G-space with base-point a_0, and let $p, q: A \to A \times A$ be given by

$$p(x) = (x, a_0), \quad q(x) = (a_0, x) \qquad (x \in A).$$

By a *Hopf G-structure* on A we mean a pointed G-map $m: A \times A \to A$ such that

$$mp \simeq 1 \simeq mq: A \to A;$$

given such a structure we refer to A as a *Hopf G-space*. For example, the reduced product space A_∞ (see [5]) of any pointed G-space A is an associative Hopf G-space [1]). If A_2 is a Hopf G-space then $\pi_G(A_1, A_2)$, for any pointed G-space A_1, obtains a natural binary operation with 0 as neutral element [2]).

If $m: A \times A \to A$ satisfies the conditions for a topological group then, as before, we describe A as a group G-space. If m satisfies these conditions up to pointed G-homotopy then we describe A as a *group-like G-space.* Note that $\pi_G(A_1, A_2)$ is a group when A_2 is group-like. This is so, in particular, when $A_2 = \Omega A_2'$ with standard Hopf G-structure for any pointed G-space A_2'. If A_2' itself is a Hopf G-space then the group is abelian, by the usual argument.

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[1]) Under suitable conditions it can be shown, using Segal's theorem, that A_∞ has the same pointed G-homotopy type as $\Omega\Sigma A$.

[2]) Another application of Segal's theorem is to show, following Sugawara [14], that this binary set forms a loop, under suitable conditions, and hence a group when the Hopf G-structure is pointed G-homotopy associative.