3. Some examples

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thus defined is bilinear and has the property that

$$[\alpha, \beta] = -(\Sigma T)^* [\beta, \alpha]. \tag{2.1}$$

It is a straightforward exercise, as in the ordinary theory, to show that the Whitehead pairing vanishes if Y is a Hopf G-space, and hence vanishes under suspension. Moreover the suspension ΣZ of a compact well-based G-space Z is a Hopf G-space if and only if the Whitehead square

$$w(\Sigma Z) \in \pi_G(\Sigma(Z \wedge Z), \Sigma Z)$$

of the identity vanishes.

It should also be noted that the Jacobi identity holds for Samelson products and hence for Whitehead products, by an equivariant version of the argument given by G. W. Whitehead [16]. Specifically, consider the permutations

$$B \wedge C \wedge A \xrightarrow{\sigma} A \wedge B \wedge C \xrightarrow{\tau} C \wedge B \wedge A,$$

where A, B, C are suspensions of pointed G-spaces. Let

$$\alpha \in \pi_G(\Sigma A, Y), \quad \beta \in \pi_G(\Sigma B, Y), \quad \gamma \in \pi_G(\Sigma C, Y),$$

where Y is a pointed G-space. Then the relation

$$[\alpha, [\beta, \gamma]] + (\Sigma\sigma)^* [\beta, [\gamma, \alpha]] + (\Sigma\tau)^* [\gamma, [\alpha, \beta]] = 0 \tag{2.2}$$

holds in the group $\pi_G(\Sigma(A \wedge B \wedge C), Y)$.

3. Some examples

We need to begin by discussing briefly some relations between the category of G-spaces and the category of pointed G-spaces, as follows. Given spaces A, B we denote points of the join $A*B$ by triples (a, b, t) where $a \in A$, $b \in B$, $t \in I$, so that (a, b, t) is independent of a when $t = 0$, of b when $t = 1$. A basepoint $b_0 \in B$ determines a basepoint $(a, b_0, 0)$ in $A*B$. If A, B are G-spaces we make $A*B$ a G-space with action

$$(a, b, t)g = (ag, bg, t) \qquad (g \in G).$$

Note that $A*B$ is pointed if B is. When $B = S^0$, with trivial action, then $A*B = \tilde{\Sigma} A$, the unreduced suspension [2]).

[1]) This differs by an automorphism from the normal definition.
[2]) We regard this as an identification space of the cylinder, in the usual way.

When A is a pointed G-space the reduced suspension ΣA is also defined and the natural projection $\tilde{\Sigma} A \rightarrow \Sigma A$ is a pointed G-homotopy equivalence if A is well-based. Moreover, if A and B are well-based the natural projection $A^*B \rightarrow \Sigma(A \wedge B)$ is also a pointed G-homotopy equivalence. We need a variant of this result.

Suppose now that $A = \tilde{\Sigma} A'$, $B = \tilde{\Sigma} B'$, where A', B' are G-spaces. Consider the pointed G-map

$$k \colon \tilde{\Sigma}(A'^*B') \rightarrow (\Sigma A' \wedge \Sigma B') = A \wedge B$$

which is given by the formula

$$k((a', b', t), s) = \begin{cases} ((a', 2st), (b', s)) & (0 \leqslant t \leqslant \tfrac{1}{2}), \\ ((a', s), (b', s(2-2t))) & (\tfrac{1}{2} \leqslant t \leqslant 1). \end{cases}$$

It is easy to check that k is a pointed G-homotopy equivalence, under similar conditions, and has the property that

$$(3.1) \qquad k \tilde{\Sigma} S = Tk,$$

where $S \colon A^*B \rightarrow B^*A$ denotes the switching map of the join and $T \colon A \wedge B \rightarrow B \wedge A$ the switching map of the smash product.

In particular, take G to be the group $O(m)$ $(m \geqslant 2)$ of orthogonal transformations of the sphere S^{m-1}. The antipodal map $a \colon S^{m-1} \rightarrow S^{m-1}$ is an $O(m)$-map, hence $\hat{a} = \tilde{\Sigma} a \colon S^m \rightarrow S^m$ is a pointed $O(m)$-map. I assert that

$$(3.2) \qquad T \simeq \hat{a} \wedge 1 \colon S^m \wedge S^m \rightarrow S^m \wedge S^m .$$

For the switching self-map S of $S^{m-1} * S^{m-1}$ is $O(m)$-homotopic to a^*1, by elementary rotation in $R^{2m} = R^m \times R^m$. Hence $\tilde{\Sigma} S$ is pointed $O(m)$-homotopic to $\tilde{\Sigma}(a^*1)$. From (3.1), therefore, we obtain that

$$Tk = k \tilde{\Sigma} S \simeq k \tilde{\Sigma}(a^*1) = (\hat{a} \wedge 1) k .$$

Since k is a pointed $O(m)$-homotopy equivalence this proves (3.2). In this case, therefore, T can be replaced by $\tilde{a} \wedge 1$ in the commutation law (2.1).

We now turn to the permutations appearing in (2.2), the Jacobi identity. More generally let A_i' $(i=1, \ldots, n)$ be a G-space. Points of the multiple join $A_1' * \ldots * A_n'$ can be represented by n-tuples of the form

$$(t_1 a_1', \ldots, t_n a_n') \qquad (a_i' \in A_i', t_i \in I)$$

where $t_1^2 + \ldots + t_n^2 = 1$. The radius vector through $(t_1, \ldots, t_n)$ meets the boundary of the n-cube I^n in a point $(x_1, \ldots, x_n)$, say, where at least one coordinate is equal to 1. Thus a pointed G-homotopy equivalence

$$l \colon \Sigma (A_1' * \ldots * A_n') \to \Sigma A_1' \wedge \ldots \wedge \Sigma A_n'$$

is given by

$$l\left((t_1 a_1', \ldots, t_n a_n'), s\right) = \left((s x_1, a_1'), \ldots, (s x_n, a_n')\right).$$

Clearly l is equivariant with respect to the action of the symmetric group on the suspension of the multiple join and on the multiple smash product.

In particular, take $G = O(m)$ and $A_i' = S^{m-1}$, for all i. Let u be a permutation of the multiple join and v the corresponding permutation of the multiple smash product. We distinguish cases according as to whether the degree of the permutation is even or odd. In the even case u is G-homotopic to the identity 1_n on the n-fold join, using elementary rotations as before, and hence v is pointed G-homotopic to the identity 1_n on the n-fold smash product. In the odd case it follows similarly that u is G-homotopic to $1_{n-1} * a$, hence v is pointed G-homotopic to $1_{n-1} \wedge \hat{a}$. Taking $n = 3$, therefore, we see that the automorphisms which appear in (2.2) are trivial, in this example, and so

$$3\left[\Sigma_* \iota_m, [\Sigma_* \iota_m, \Sigma_* \iota_m]\right] = 0 \tag{3.3}$$

in $\pi_G(S^{3m+1}, S^{m+1})$, where ι_m denotes the pointed $O(m)$-homotopy class of the identity on S^m. It is easy to see, incidentally, that the Whitehead square $[\Sigma_* \iota_m, \Sigma_* \iota_m] \in \pi_G(S^{2m+1}, S^{m+1})$ is of infinite order, for all $m \geqslant 2$.

4. Ex-homotopy theory

For our second example of an alternative homotopy theory we take the category of ex-spaces (see [7] for details), which is an enlargement of the category of sectioned bundles mentioned earlier. We recall that, with regard to a given space X, an *ex-space* consists of a space E together with maps

$$X \xrightarrow{\sigma} E \xrightarrow{\rho} X$$

such that $\rho\sigma = 1$. We refer to ρ as the *projection*, to σ as the *section*, and to (ρ, σ) as the *ex-structure*. Let E_i $(i = 1, 2)$ be an ex-space with ex-structure