

# 6. The exact sequence

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **23 (1977)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **26.05.2024**

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and so  $k^{q+1}s \simeq k^{q+1}t$  over  $X^n$ . Hence, by induction, we obtain (5.1). Of course the value of  $r$  can often be improved in particular cases.

**COROLLARY (5.2).** *Let  $E$  be a fibre bundle over  $X$  with locally compact fibre  $F$ , which admits a cross-section. Choose a cross-section and so regard  $E$  as an ex-space. Let  $f: E \rightarrow E$  be an ex-map such that  $g: F \rightarrow F$  is null-homotopic, where  $g = f|_F$ . Then  $f^{r+1} \simeq c$ , the trivial ex-map, where  $r = \text{reg}(X)$ .*

To see this, take  $M = M_X(E, E)$ , in (5.1), and define  $k: M \rightarrow M$  by post-composition with  $f$ . We take  $s, t$  to be the cross-sections  $f', e': X \rightarrow M$  determined by  $f, e$ , and obtain (5.2).

Now let  $\alpha, \beta \in \pi_X(E, E)$  be elements such that

- (i)  $\alpha^2 = \beta^2$  and  $\alpha\beta = \beta\alpha$ ,
- (ii)  $\Phi_*\alpha = \Phi_*\beta$ ,

where  $\Phi_*: \pi_X(E, E) \rightarrow \pi(F, F)$  is given by restriction. Suppose that  $E = \Sigma E'$ , for some ex-space  $E'$ , and that  $\alpha = \Sigma_*\alpha'$ ,  $\beta = \Sigma_*\beta'$ , for some  $\alpha' \in \pi_{X'}(E', E')$ . Take  $f$  in (5.2) to be a representative of  $\alpha - \beta$ . Then  $f^{r+1}$  is a representative of  $(\alpha - \beta)^{r+1} = 2^r(\alpha - \beta)\alpha^r$ , and so (5.2) shows that

$$(5.3) \quad 2^r\alpha = 2^r\beta.$$

Applications will be given in §8 below.

## 6. THE EXACT SEQUENCE

Let  $X$  be a CW-complex with basepoint  $x_0$  a 0-cell. Let  $p: M \rightarrow X$  be a fibration with fibre  $N = p^{-1}(x_0)$ , and let  $\Gamma$  denote the function-space of cross-sections. By evaluating at  $x_0$  we obtain a fibration  $q: \Gamma \rightarrow N$ . It may be noted that, under fairly general conditions, this fibration admits a cross-section if and only if the original fibration is trivial, in the sense of fibre homotopy type.

Now choose a basepoint  $y_0 \in N$  so that  $q^{-1}(y_0) = \Gamma_0$ , the space of pointed cross-sections. Choose such a cross-section  $s$  as basepoint in  $\Gamma_0 \subset \Gamma$ , and consider the homotopy exact sequence of the fibration as follows:

$$\dots \rightarrow \pi_{r+1}(N) \xrightarrow{\Delta} \pi_r(\Gamma_0) \xrightarrow{u_*} \pi_r(\Gamma) \xrightarrow{q_*} \pi_r(N) \rightarrow \dots$$

Note that  $\Gamma_0$  is a deformation retract of  $\tilde{\Gamma}_0$ , the space of pointed maps  $t: X \rightarrow M$  such that  $pt \simeq 1$ .

In particular, suppose that  $X = S^n$  ( $n \geq 1$ ). Then  $s$  determines a homotopy equivalence  $k: \Omega^n(N) \rightarrow \Gamma_0$  as follows. Consider the map  $l: \Omega^n(N) \rightarrow \tilde{\Gamma}_0$  which transforms each pointed map  $f: S^n \rightarrow N$  into the track sum  $s + j f$ , where  $j: N \subset M$ . Then  $l$  is a homotopy equivalence and  $k$  is obtained by composing  $l$  with a deformation retraction  $\tilde{\Gamma}_0 \rightarrow \Gamma$ . In this case, therefore, we can transform our exact sequence into

$$\dots \rightarrow \pi_{r+1}(N) \xrightarrow{D} \pi_{r+n}(N) \xrightarrow{v_*} \pi_r(\Gamma) \xrightarrow{q_*} \pi_r(N) \rightarrow \dots,$$

where  $v_* = u_* k_*$  and  $D = k_*^{-1} \Delta$ . We refer to this as the *modified exact sequence* of the evaluation fibration. The operator  $D$  has been determined in §3 of [8]. Specifically, let  $\sigma \in \pi_n(M)$  denote the class of  $s$ , and let  $\alpha \in \pi_{r+1}(N)$ . Then

$$(6.1) \quad j_* D\alpha = [j_* \alpha, \sigma],$$

the Whitehead product in  $\pi_*(M)$ .

We shall be particularly concerned with the tail end of this sequence, which reads

$$\pi_1(N) \xrightarrow{D} \pi_n(N) \xrightarrow{v_*} \pi(\Gamma) \xrightarrow{q_*} \pi(N).$$

Let  $t: S^n \rightarrow M$  also be a cross-section. Suppose that we have a path  $\lambda$  in  $N$  from  $s(x_0)$  to  $t(x_0)$ . We can regard  $\lambda$  as a vertical homotopy of  $s$  into  $t$  over  $\{x_0\}$ . The obstruction to extending this to a vertical homotopy over  $S^n$  is an element

$$\delta(s, t; \lambda) \in \pi_1(N).$$

If  $\mu$  is another path in  $N$  from  $s(x_0)$  to  $t(x_0)$  the track difference  $\lambda - \mu$  forms a loop in  $N$  and it is easy to check that the homotopy class  $\alpha \in \pi_1(N)$  of this loop satisfies the relation

$$D\alpha = \delta(s, t; \lambda) - \delta(s, t; \mu).$$

Hence  $s$  and  $t$  are vertically homotopic if and only if the obstruction is contained in  $D\pi_1(N)$ .

Now let  $E_i$  ( $i=1, 2$ ) be a sectioned bundle over  $X$  with locally compact fibre  $F_i$ . We can apply the above to the function-space bundle  $M = M_X(E_1, E_2)$  with fibre  $N = N(F_1, F_2)$ , and obtain useful information about the ex-homotopy groups  $\pi_X(\Sigma^r E_1, E_2)$  ( $r=1, 2, \dots$ ). Details are given in [9] where the operator  $D$  is calculated, as follows, in case  $E_1$  and  $E_2$  are sphere-bundles over  $X = S^n$ .

Given a representation  $\phi: SO(m) \rightarrow SO(q)$  write

$$J_\phi = J \circ \phi_*: \pi_r SO(m) \rightarrow \pi_{r+q}(S^q),$$

where  $J$  denotes the usual Hopf-Whitehead homomorphism. For example, if  $q > m$  and  $\phi$  is the inclusion then

$$(6.2) \quad J_\phi = (-1)^{m-q} \Sigma_*^{m-q} J,$$

by (3.2) of [5] (cf. [8]). If  $q = 2m$  and  $\phi = 1 \oplus 1$  it is easily seen that

$$(6.3) \quad J_\phi = 2(-1)^m \Sigma_*^m J.$$

Consider the function-space  $N = N(S^p, S^q)$  of pointed maps  $S^p \rightarrow S^q$ . We identify  $\pi_i(N)$  ( $i=0, 1, \dots$ ) with  $\pi_{i+p}(S^q)$  in the standard way (see [15]). Let  $G$  be a topological group and let

$$\phi: G \rightarrow SO(p), \quad \psi: G \rightarrow SO(q)$$

be representations of  $G$ . We regard  $S^p, S^q$  as pointed  $G$ -spaces using  $\phi, \psi$ , respectively. Choose a principal  $G$ -bundle  $P$  over  $S^n$  with classifying element  $\theta \in \pi_{n-1}(G)$ , and take  $E_1 = P_{\#}(S^p)$ ,  $E_2 = P_{\#}(S^q)$ . Then the operator  $D$  in our exact sequence is given

$$(6.4) \quad D\alpha = \alpha \circ \Sigma_*^{r+p-q+1} J_\psi \theta - J_\phi \theta \circ \Sigma_*^{n+p-q-1} \alpha,$$

where  $\alpha \in \pi_{r+p+1}(S^q)$ . The case  $r = 1$  of this result will be needed in §8 below.

## 7. THE ADJOINT $G$ -BUNDLE

Let  $X$  be any space and let  $P$  be a principal  $G$ -bundle over  $X$ . We regard  $P$  as a (right)  $G$ -space in the usual way. By a *principal automorphism* we mean an equivariant fibre-preserving map of  $P$  into itself. By the *adjoint  $G$ -bundle* we mean the sectioned bundle  $Q = P_{\#}G$ , where  $G$  acts on itself by conjugation. Note that  $Q$  is a group ex-space since  $G$  is a group  $G$ -space. We can construct  $Q$  from  $G \times P$  by identifying

$$(7.1) \quad (gag^{-1}, b) \sim (a, bg) \quad (a \in G, b \in P)$$

for all  $g \in G$ . The group ex-structure is given by

$$\{a_1, b\} \cdot \{a_2, b\} = \{a_1 \cdot a_2, b\} \quad (a_1, a_2 \in G),$$

where  $\{ , \}$  denotes the equivalence class of  $( , )$ . Every principal automorphism  $f$  of  $P$  determines a cross-section  $f': X \rightarrow Q$  as follows.