

7. The adjoint G-bundle

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **23 (1977)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **05.06.2024**

Nutzungsbedingungen

Die ETH-Bibliothek ist Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Inhalten der Zeitschriften. Die Rechte liegen in der Regel bei den Herausgebern.

Die auf der Plattform e-periodica veröffentlichten Dokumente stehen für nicht-kommerzielle Zwecke in Lehre und Forschung sowie für die private Nutzung frei zur Verfügung. Einzelne Dateien oder Ausdrucke aus diesem Angebot können zusammen mit diesen Nutzungsbedingungen und den korrekten Herkunftsbezeichnungen weitergegeben werden.

Das Veröffentlichen von Bildern in Print- und Online-Publikationen ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Die systematische Speicherung von Teilen des elektronischen Angebots auf anderen Servern bedarf ebenfalls des schriftlichen Einverständnisses der Rechteinhaber.

Haftungsausschluss

Alle Angaben erfolgen ohne Gewähr für Vollständigkeit oder Richtigkeit. Es wird keine Haftung übernommen für Schäden durch die Verwendung von Informationen aus diesem Online-Angebot oder durch das Fehlen von Informationen. Dies gilt auch für Inhalte Dritter, die über dieses Angebot zugänglich sind.

Given a representation $\phi: SO(m) \rightarrow SO(q)$ write

$$J_\phi = J \circ \phi_*: \pi_r SO(m) \rightarrow \pi_{r+q}(S^q),$$

where J denotes the usual Hopf-Whitehead homomorphism. For example, if $q > m$ and ϕ is the inclusion then

$$(6.2) \quad J_\phi = (-1)^{m-q} \Sigma_*^{m-q} J,$$

by (3.2) of [5] (cf. [8]). If $q = 2m$ and $\phi = 1 \oplus 1$ it is easily seen that

$$(6.3) \quad J_\phi = 2(-1)^m \Sigma_*^m J.$$

Consider the function-space $N = N(S^p, S^q)$ of pointed maps $S^p \rightarrow S^q$. We identify $\pi_i(N)$ ($i=0, 1, \dots$) with $\pi_{i+p}(S^q)$ in the standard way (see [15]). Let G be a topological group and let

$$\phi: G \rightarrow SO(p), \quad \psi: G \rightarrow SO(q)$$

be representations of G . We regard S^p, S^q as pointed G -spaces using ϕ, ψ , respectively. Choose a principal G -bundle P over S^n with classifying element $\theta \in \pi_{n-1}(G)$, and take $E_1 = P_{\#}(S^p)$, $E_2 = P_{\#}(S^q)$. Then the operator D in our exact sequence is given

$$(6.4) \quad D\alpha = \alpha \circ \Sigma_*^{r+p-q+1} J_\psi \theta - J_\phi \theta \circ \Sigma_*^{n+p-q-1} \alpha,$$

where $\alpha \in \pi_{r+p+1}(S^q)$. The case $r = 1$ of this result will be needed in §8 below.

7. THE ADJOINT G -BUNDLE

Let X be any space and let P be a principal G -bundle over X . We regard P as a (right) G -space in the usual way. By a *principal automorphism* we mean an equivariant fibre-preserving map of P into itself. By the *adjoint G -bundle* we mean the sectioned bundle $Q = P_{\#}G$, where G acts on itself by conjugation. Note that Q is a group ex-space since G is a group G -space. We can construct Q from $G \times P$ by identifying

$$(7.1) \quad (gag^{-1}, b) \sim (a, bg) \quad (a \in G, b \in P)$$

for all $g \in G$. The group ex-structure is given by

$$\{a_1, b\} \cdot \{a_2, b\} = \{a_1 \cdot a_2, b\} \quad (a_1, a_2 \in G),$$

where $\{ , \}$ denotes the equivalence class of $(,)$. Every principal automorphism f of P determines a cross-section $f': X \rightarrow Q$ as follows.

Given $x \in X$ choose any $b \in P_x$; then $fb = bg$, for some $g \in G$, and we define $f'x = \{g, b\}$. This correspondence establishes an isomorphism between the group of principal automorphisms of P and the group of cross-sections of Q .

Any element c of the centre of G determines a G -map $c_{\#}$ for any G -space A . Notice that $c_{\#}$ is a principal automorphism in the case of P and that the corresponding cross-section $c'_{\#}$ of Q is given by $c'_{\#}\{b\} = \{c, b\}$. When X is a sphere these central cross-sections of Q can be analysed as follows.

Take $X = S^n$ ($n \geq 2$), so that P is a principal G -bundle over S^n . Let B^n denote the n -ball with boundary S^{n-1} . Choose a relative homeomorphism $(B^n, S^{n-1}) \rightarrow (S^n, x_0)$ and lift this to a map $k: (B^n, S^{n-1}) \rightarrow (P, G)$. The homotopy class $\theta \in \pi_{n-1}(G)$ of $l = k|S^{n-1}$ classifies the bundle according to clutching theory.

Let $c \in G$ be central and let $\lambda: I \rightarrow G$ be a path such that $\lambda(0) = e$, $\lambda(1) = c$. Consider the map $\Lambda: B^n \times I \rightarrow Q$ which is given by

$$\Lambda(y, t) = \{\lambda(t), k(y)\} \quad (y \in B^n, t \in I).$$

The boundary of $B^n \times I$ is the sphere

$$B^n \times 0 \cup S^{n-1} \times I \cup B^n \times 1,$$

and Λ maps $S^{n-1} \times I$ into $G \subset Q$ by

$$\Lambda(y, t) = (ly) \cdot (\lambda t) \cdot (ly)^{-1},$$

using (7.1). Let us compare this with the map Λ' of the boundary which agrees with Λ on $B^n \times I$ but is given on $S^{n-1} \times I$ by $\Lambda'(y, t) = \lambda t$. Now λ can be regarded as a vertical homotopy of $e'_{\#}$ into $c'_{\#}$ over $\{x_0\}$ and Λ represents the obstruction

$$\delta = \delta(e'_{\#}, c'_{\#}; \lambda) \in \pi_n(G)$$

to extending this vertical homotopy over S^n . Since $\Lambda|_{(B^n \times I)}$ is null-homotopic, however, it follows that δ is also represented by $d: \tilde{\Sigma} S^{n-1} \rightarrow G$, where

$$d(y, t) = (ly) \cdot (\lambda t) \cdot (ly)^{-1}.$$

For example, take $G = SO(m)$, with m even. Take $c = -e$ and

$$\lambda(t) = e \cos \pi t + b \sin \pi t \quad (0 \leq t \leq 1),$$

where b denotes the matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \oplus \dots \oplus \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad (m/2 \text{ summands}).$$

Then $\delta = F\theta$, by definition, where

$$F: \pi_{n-1} SO(m) \rightarrow \pi_n SO(m)$$

denotes the Bott suspension, as in [6].

Now let A_i ($i=1, 2$) be a locally compact pointed G -space and write $E_i = P_\# A_i$. Recall that $N = N(A_1, A_2)$ denotes the function-space of pointed maps $A_1 \rightarrow A_2$. Given a pointed G -map $f: A_1 \rightarrow A_2$ we can construct an ex-map $P_\# f: E_1 \rightarrow E_2$ and a pointed G -map $\bar{f}: G \rightarrow N$, where $\bar{f}(g) = g_\# \circ f = f \circ g_\#$. I assert

PROPOSITION (7.2). *The ex-maps*

$$P_\# f, P_\# f \circ P_\# c: E_1 \rightarrow E_2$$

are ex-homotopic if and only if

$$\bar{f}^* \delta \in D\pi_1(N) \subset \pi_n(N),$$

where δ is as above.

Here D is the operator which occurs in the modified exact sequence of the evaluation fibration derived from the function-space bundle, as in §6. The proof of (7.2) is by naturality, as follows.

First observe that \bar{f} extends to a fibre-preserving map $\hat{f}: Q \rightarrow M$, where $M = M_X(E_1, E_2)$ denotes the function-space bundle. To see this we note that f determines a pointed G -map $F: A_1 \times G \rightarrow A_2$, where

$$F(x, g) = f(xg) \quad (x \in A_1, g \in G).$$

Hence $P_\# f: E_1 \times Q \rightarrow E_2$ is defined and we take \hat{f} to be the adjoint.

We have $X = S^n$ so that the evaluation fibrations can be modified as in §6. Clearly

$$(7.3) \quad \Gamma_0(\hat{f}) \circ k \simeq l \circ \Omega^n(\bar{f})$$

as shown below, where k is defined by subtracting the cross-section $e' \#$ and l by subtracting $\hat{f} \circ e' \#$.

$$\begin{array}{ccc} \Omega^n(G) & \xrightarrow{k} & \Gamma_0(Q) \\ \Omega^n(\bar{f}) \downarrow & & \downarrow \Gamma_0(\hat{f}) \\ \Omega^n(N) & \xrightarrow{l} & \Gamma_0(M) \end{array}$$

Hence we obtain a commutative diagram as follows, relating the modified exact sequences for Q and M .

$$\begin{array}{ccc} \pi_n(G) & \xrightarrow{u_*} & \pi(\Gamma(Q)) \\ \bar{f}_* \downarrow & & \downarrow (\hat{\Gamma(f)})_* \\ \pi_n(N) & \xrightarrow{v_*} & \pi(\Gamma(M)) \end{array}$$

Recall that δ is the obstruction to extending λ to a vertical homotopy of $e'_{\#}$ into $c'_{\#}$. Hence $\bar{f}_*\delta$ is the obstruction to extending $\bar{f} \circ \lambda$ to a vertical homotopy of $\hat{f} \circ e'_{\#}$ into $\hat{f} \circ c'_{\#}$. Hence it follows, as explained in the previous section, that $\hat{f} \circ e'_{\#}$ and $\hat{f} \circ c'_{\#}$ are vertically homotopic if and only if $\delta \in D\pi_1(N)$. Finally we use the correspondence between ex-maps and cross-sections to obtain (7.2) as stated.

8. EXAMPLES

Let X be a finite simply-connected complex and let P be a principal $SO(m)$ -bundle over X . Consider the antipodal self-map a of S^{m-1} . The unreduced suspension \hat{a} is a pointed $SO(m)$ -map of S^m into itself. Hence $P_{\#}\hat{a}$ is an ex-map of $E = P_{\#}S^m$ into itself; let $\sigma \in \pi_X(E, E)$ denote the ex-homotopy class. Since \hat{a} is of degree $(-1)^m$ we can apply (5.3) and obtain that

$$(8.1) \quad 2^r \Sigma_* \sigma = 2^r \quad (m \text{ even}),$$

where $r = \text{reg}(X)$. It follows at once that

$$(8.2) \quad 2^{r+1} [\iota_{\Sigma E}, \iota_{\Sigma E}] = 0 \quad (m \text{ even}),$$

by (2.1) and (3.1), and hence from (3.3) that

$$(8.3) \quad [\iota_{\Sigma E}, [\iota_{\Sigma E}, \iota_{\Sigma E}]] = 0 \quad (m \text{ even}).$$

Here $\iota_{\Sigma E}$ denotes the ex-homotopy class of the identity on ΣE . Similar results, but under more restrictive conditions, have been obtained by Eggar [4]. It can also be shown that the quadruple Whitehead products

$$[[[\iota_{\Sigma E}, \iota_{\Sigma E}], [\iota_{\Sigma E}, \iota_{\Sigma E}]], [\iota_{\Sigma E}, [\iota_{\Sigma E}, [\iota_{\Sigma E}, \iota_{\Sigma E}]]]]$$

are trivial, whether m is even or odd.