8. Examples

Hence we obtain a commutative diagram as follows, relating the modified exact sequences for Q and M.

$$\begin{array}{ccc} \pi_n(G) & \xrightarrow{u_*} & \pi(\Gamma(Q)) \\ \bar{f}_* \downarrow & & \downarrow (\Gamma(\hat{f}))_* \\ \pi_n(N) & \xrightarrow[v_*]{} & \pi(\Gamma(M)) \end{array}$$

Recall that δ is the obstruction to extending λ to a vertical homotopy of $e'_{\#}$ into $c'_{\#}$. Hence $\bar{f}_*\delta$ is the obstruction to extending $\bar{f} \circ \lambda$ to a vertical homotopy of $\hat{f} \circ e'_{\#}$ into $\hat{f} \circ c'_{\#}$. Hence it follows, as explained in the previous section, that $\hat{f} \circ e'_{\#}$ and $\hat{f} \circ c'_{\#}$ are vertically homotopic if and only if $\delta \in D\pi_1(N)$. Finally we use the correspondence between ex-maps and cross-sections to obtain (7.2) as stated.

8. Examples

Let X be a finite simply-connected complex and let P be a principal $SO(m)$-bundle over X. Consider the antipodal self-map a of S^{m-1}. The unreduced suspension $\hat{a}$ is a pointed $SO(m)$-map of S^m into itself. Hence $P_{\#}\hat{a}$ is an ex-map of $E = P_{\#}S^m$ into itself; let $\sigma \in \pi_X(E, E)$ denote the ex-homotopy class. Since $\hat{a}$ is of degree $(-1)^m$ we can apply (5.3) and obtain that

$$2^r \Sigma_* \sigma = 2^r \qquad (m \text{ even}), \tag{8.1}$$

where $r = \operatorname{reg}(X)$. It follows at once that

$$2^{r+1}[\iota_{\Sigma E}, \iota_{\Sigma E}] = 0 \qquad (m \text{ even}), \tag{8.2}$$

by (2.1) and (3.1), and hence from (3.3) that

$$[\iota_{\Sigma E}, [\iota_{\Sigma E}, \iota_{\Sigma E}]] = 0 \qquad (m \text{ even}). \tag{8.3}$$

Here $\iota_{\Sigma E}$ denotes the ex-homotopy class of the identity on ΣE. Similar results, but under more restrictive conditions, have been obtained by Eggar [4]. It can also be shown that the quadruple Whitehead products

$$[[\iota_{\Sigma E}, \iota_{\Sigma E}], [\iota_{\Sigma E}, \iota_{\Sigma E}]], [\iota_{\Sigma E}, [\iota_{\Sigma E}, [\iota_{\Sigma E}, \iota_{\Sigma E}]]]$$

are trivial, whether m is even or odd.

In particular, let X be a sphere. For m even (8.1) shows that $2\Sigma_*\sigma = 2$ and (8.2) that $4[\iota_{\Sigma E}, \iota_{\Sigma E}] = 0$. However, more precise results can be obtained by using the methods of §7, as follows. Take $X = S^n$ $(n \geqslant 2)$ and let $\theta \in \pi_{n-1} SO(m)$ be the classifying element of P. We apply (7.2) with $f = 1$ and, using (6.4), obtain

THEOREM (8.4). *Let m be even. Then $\Sigma_*\sigma = 1$ if and only if $\Sigma_* JF\theta$ is contained in the image of*

$$D: \pi_{m+2}(S^{m+1}) \to \pi_{n+m+1}(S^{m+1}),$$

where

$$D\alpha = \Sigma_* J\theta \circ \Sigma_*^{n-1}\alpha - \alpha \circ \Sigma_*^2 J\theta .$$

In the stable range, where $m > n$, the homomorphism D is trivial and $F\theta = \theta \circ \eta$, as in §6 of [6], where η generates the 1-stem. In this range it does not matter whether we deal with ex-maps or over-maps, and so (8.4) agrees with (4.5) of [8].

Now let ι_m denote the pointed $SO(m)$-homotopy class of the identity on S^m, so that $\iota_{\Sigma E} = P_{\#}\Sigma_*\iota_m$. Represent the Whitehead square

$$w(\Sigma S^m) = [\Sigma_*\iota_m, \Sigma_*\iota_m]$$

by a pointed $SO(m)$-map $f: \Sigma(S^m \wedge S^m) \to \Sigma S^m$. Then $P_{\#}f$ represents the Whitehead square

$$w(\Sigma E) = P_{\#}w(\Sigma S^m) = [\iota_{\Sigma E}, \iota_{\Sigma E}] .$$

We apply (7.2) again and, using (6.2)-(6.4), obtain

THEOREM (8.5). *Let m be even. Then $2w(\Sigma E) = 0$ if and only if $w_{m+1} \circ \Sigma_*^{m+1} JF\theta$ lies in the image of*

$$D: \pi_{2m+2}(S^{m+1}) \to \pi_{n+2m+1}(S^{m+1}),$$

where $D\alpha = \alpha \circ \Sigma_*^{m+2} J\theta - 2\Sigma^* J\theta \circ \Sigma_*^{n-1}\alpha$.

Here $w_{m+1} \in \pi_{2m+1}(S^{m+1})$ denotes the ordinary Whitehead square of the generator of $\pi_{m+1}(S^{m+1})$. Unless $m = 2$ or 6 we have $w_{m+1} \neq 0$ and (8.5) determines the order of $w(\Sigma E)$. When $m = 2$ or 6 it would be interesting to know when $w(\Sigma E) = 0$, i.e. when ΣE is a Hopf ex-space, but unfortunately our method does not apply.

For some examples where the order of $w(\Sigma E)$ is (precisely) 4, consider the transgression $\Delta: \pi_n(S^m) \to \pi_{n-1} SO(m)$ in the homotopy exact sequence of the standard fibration of $SO(m+1)$. Take $\theta = \Delta\phi$, where $\phi \in \Sigma_* \pi_{n-1}(S^{m-1})$. Then $\Sigma_* J\theta = 0$ and so D is trivial. However it follows from (4.1) and (6.3) of [6] that

$$\Sigma_* JF\theta = [\Sigma_* \iota_m, \Sigma_* \phi].$$

This Whitehead product is non-zero if, for example, $m = n$ and $\phi = \iota_m$ with $m \neq 2, 6$. Of course E is trivial as a bundle, in these examples, although not as a sectioned bundle.

REFERENCES

[1] ARKOWITZ, M. The generalized Whitehead product. *Pacific J. of Math.* 12 (1962), 7-23.
[2] BECKER, J. C. On the existence of A_k-structure on stable vector bundles. *Topology 9* (1970), 367-384.
[3] COHEN, D. E. Products and carrier theory. *Proc. London Math. Soc. (3) 7* (1957), 219-248.
[4] EGGAR, M. H. D. Phil. thesis (Oxford 1970).
[5] JAMES, I. M. Reduced product spaces. *Ann. of Math. 62* (1955), 170-197.
[6] —— On the Bott suspension. *J. of Math. Kyoto Univ. 9* (1969), 161-188.
[7] —— Ex-homotopy theory. *Illinois J. of Math. 15* (1971), 324-337.
[8] —— On sphere-bundles with certain properties. *Quart. J. of Math. Oxford (2), 22* (1971), 353-370.
[9] —— On the maps of one fibre space into another. *Comp. Math. 23* (1971), 317-328.
[10] —— Products between homotopy groups. *Comp. Math. 23* (1971), 329-345.
[11] NOAKES, J. L. D. Phil. thesis (Oxford 1974).
[12] PUPPE, D. Homotopiemengen und ihre induzierten abbildungen I. *Math. Zeitschr. 69* (1958), 299-344.
[13] SEGAL, G. B. and JAMES, I. M. Note on equivariant homotopy type *(to appear)*.
[14] SUGAWARA, M. H-spaces and spaces of loops. *Math. J. Okayama Univ. 5* (1955), 5-11.
[15] WHITEHEAD, G. W. On products in homotopy groups. *Ann. of Math. 47* (1946), 460-475.
[16] —— On mappings into group-like spaces. *Comm. Math. Helv. 28* (1954), 320-328.
[17] WHITEHEAD, J. H. C. On the homotopy type of ANR's. *Bull. Amer. Math. Soc. 54* (1958), 1125-1132.
[18] —— On certain theorems of G. W. Whitehead. *Ann. of Math. 58* (1953), 418-428.
[19] WILSON, G. S0(3)-equivariant maps of spheres. *Quart. J. of Math. Oxford (2), 27* (1976), 263-265.

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