4. NON-EUCLIDEAN MOTIONS

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4. Non-euclidean motions

The euclidean case was dealt with in [3]. In the present paper we undertake a more detailed study of the hyperbolic case. The unit ball in $\mathbf{R}^n$ is denoted by B, and G is the full group of Möbius transformations mapping B on itself. The Poincaré metric $ds = (1-|x|^2)^{-1} \, |dx|$ and the non-euclidean volume element $\rho \, dx = (1-|x|^2)^{-n} \, dx$ are invariant under G.

For $A \in G$ we prefer to denote the Jacobian by $A'(x)$ rather than $DA(x)$. We use $|A'(x)|$ for the linear rate of change, the same in all directions. This notation has the advantage of leading to formulas which are easily recognizable generalizations of the familiar formulas for $n = 2$ in complex notation. $|A'(x)|$ is also the square norm of the matrix $A'(x)$, and $|\det A'(x)| = |A'(x)|^n$.

Reflection in the unit sphere is denoted by $x^* = x/|x|^2$. Its Jacobian is $Dx^* = |x|^{-2}(1_n - 2Q(x))$ with $Q(x)_{ij} = x_i x_j / |x|^2$; note that $(1_n - 2\,Q(x))^2 = 1_n$.

For every $y \in B$ there is a unique $T_y \in G$ such that $T_y y = 0$ and $T'_y(y) = |T'_y(y)| \cdot 1_n$. The most general $A \in G$ is of the form $A = UT_y$ with $y = A^{-1}(0)$ and $U \in O(n)$.

For $n = 2$, in complex notation,

$$T_y x = \frac{x - y}{1 - \bar{y}x}$$

$$T'_y(x) = \frac{1 - |y|^2}{(1 - \bar{y}x)^2} \, .$$

The first formula can be rewritten as

$$T_y x = \frac{(x-y)(1 - |y|^2) - |x-y|^2 \, y}{|y|^2 \, |x - y^*|^2} \, .$$

In this form it makes sense for arbitrary n and is in fact the correct formula. The denominator $|y|^2 \, |x - y^*|^2$ corresponds to $|1 - \bar{y}x|^2$, and it is equal to $1 - 2(xy) + |x|^2 \, |y|^2$, where (xy) is the inner product. To emphasize the symmetry we shall use the notation $|y| \, |x - y^*| = |x| \, |y - x^*| = [x, y]$.

The expression for $T'_y(x)$ is

$$T'_y(x) = \frac{1 - |y|^2}{[x, y]^2} \, \Delta(x, y)$$

with

$$\Delta(x, y) = (1-2Q(y))(1-2Q(x-y^*)) = (1-2Q(y-x^*))(1-2Q(x)) .$$

Observe that $\Delta(x, y) = {}^t\Delta(y, x)$ and $\Delta(x, y)^2 = 1_n$ so that $\Delta(x, y) \in O(n)$. The matrix $\Delta(x, y)$ generalizes the angle arg $(1-\bar{x}y)/(1-\bar{y}x)$.

It is useful to note that $|Ax - Ay|^2 = |A'(x)|\,|A'(y)|\,|x-y|^2$ for any Möbius transformation A, and $[Ax, Ay]^2 = |A'(x)|\,|A'(y)|\,[x, y]^2$ if $A \in G$. There is an important relation between $T_y x$ and $T_x y$ expressed by

$$T_y x = -\Delta(x, y)\, T_x y . \tag{4}$$

We refer to [2, 3, 4, 5] for the elementary proofs of these formulas.

5. Fundamental solutions

A continuous mapping $f : B \to \mathbf{R}^n$ will be called a *deformation*. In this paper we shall assume, mainly for simplicity, that f is continuous on the boundary $S(1)$, and that $x \cdot f(x) = 0$ on $S(1)$; this means that f maps B on itself when regarded as an infinitesimal mapping.

A deformation is *trivial* if $Sf = 0$. There are very few trivial deformations: a complete list is given in [3].

It is customary to say that f is a *quasiconformal* deformation if $\|Sf\| \in L^\infty(B)$; here $\|Sf\|$ is the function whose value at x is the square norm of the matrix $Sf(x)$. More generally, we shall also consider functions with $\|Sf\| \in L^p(B)$; we abbreviate to $Sf \in L^p$, and we denote the L^p-norm of the square norm by $\|Sf\|_p$. The same convention will prevail for all matrix-valued functions.

We shall say that f is *harmonic* if $S^* \rho\, Sf = 0$, $\rho = (1-|x|^2)^{-n}$. Because of the invariance, if f is harmonic and $A \in G$, then $A^* f$ is also harmonic. Harmonicity in this sense is not the same as requiring the components to be harmonic with respect to the Poincaré metric.

There are n linearly independent solutions of the equation $S^* \gamma = 0$ which are homogeneous of degree $1-n$. We denote them by $\gamma_{..,k}$, $k = 1, \dots, n$, the elements being

$$\gamma_{ij,k}(x) = |x|^{-n}(\delta_{ik}x_j + \delta_{jk}x_i - \delta_{ij}x_k) + (n-2)\,|x|^{-n-2}\, x_i x_j x_k .$$

There is a unique vector-valued function $g_{.k}(x)$ with components $g_{ik}(x)$ such that $g_{.k}(x) = 0$ for $|x| = 1$ and $\rho\, Sg_{.k} = \gamma_{..,k}$ so that