2. The Computation of the Degrees for Real Finite Reflection Groups

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3) *Each* G_i, $1 \leqslant i \leqslant k$, *is one of the groups described in Theorem 3.5.* G *is a Coxeter group iff* $V_0 = 0$.

The proof of Theorem 3.6 is identical with that of Theorem 2.7. We simply observe that we may now choose the V_i's to be mutually orthogonal.

2. The Computation of the Degrees for Real Finite Reflection Groups

Let G be a finite irreducible orthogonal reflection group acting on the n-dimensional Euclidean space R^n. Let F be a fundamental region as described in Theorem 3.3 and $R_1, \ldots, R_n$ the n reflections in the walls of F. We shall relate the degrees $d_1, \ldots, d_n$ of the basic homogeneous invariants to the eigenvalues of $R_1 \ldots R_n$. We first prove

Theorem 3.7. *Let* $\sigma(i)$ *be any permutation of* $1, \ldots, n$. *Then* $R_1 \ldots R_n$ *is conjugate to* $R_{\sigma(1)} \ldots R_{\sigma(n)}$

Proof. Observe that $R_1 (R_1 \ldots R_n) R_1 = R_2 \ldots R_n R_1$ so that all cyclic permutations yield conjugate transformations. We may also permute any two adjacent R_i's for which the corresponding walls are orthogonal, as the R_i's then commute. Theorem 3.7 will then follow from the following

Lemma 3.1. Let $p_1, \ldots, p_n$ be nodes of a tree T. Any circular arrangement of $1, \ldots, n$ can be obtained from a sequence of interchanges of pairs i, j which are adjacent on the circle and for which p_i, p_j are not linked in T.

Proof of Lemma 3.1. We proceed by induction, the result being obvious for $n = 1$ or 2. We may assume that p_n is an end node of the tree, i.e. it links to precisely one other node. We first rearrange $1, \ldots, n-1$ as we wish. To show that this can be done, we just consider the possibility - - - *inj* - - - where p_i, p_j are not linked. If p_i, p_n are not linked, then we interchange first i, n and then i, j, obtaining - - - *nji* - - -. If p_j, p_n are not linked, then we first interchange j, n and then j, i, obtaining - - - *jin* - - -. We may therefore arrange $1, \ldots, n-1$ in the desired order. Shifting n in one direction, which is permissible as n just fails to commute with one element, we obtain the desired arrangement of $1, \ldots, n$.

In view of Theorem 3.7, the eigenvalues of $R_1 \ldots R_n$ are independent of the order in which the R_i's appear. They are also independent of the particularly chosen F. For let F' be another fundamental region as described in Theorem 3.3. Then $F' = \sigma F$, $\sigma \in G$. The reflections in the walls of F'

are given by $R_i' = \sigma R_i \sigma^{-1}$, $1 \leqslant i \leqslant n$, so that $R_1' \dots R_n' = \sigma R_1 \dots R_n \sigma^{-1}$. The main result of the present section is the following

THEOREM 3.8 (Coleman [8]). *Let* $R_1 \dots R_n$ *have order* h. *Let* $\zeta = e^{2\pi i/h}$. *The eigenvalues of* $R_1 \dots R_n$ *are given by* $\zeta^{(d_j - 1)}$, $1 \leqslant j \leqslant n$, *the* $d_j' s$ *being the degrees of the basic homogeneous invariants of* G.

Theorem 3.8. was first obtained by Coxeter [7], who verified this fact for each group listed in Theorem 3.5. Coleman [8] supplied a general proof, using the fact that the number of reflections $= \frac{1}{2} nh$. This fact, which was at first known only by individual verification [7], was proven by Steinberg [20]. In view of Theorem 3.8, the numbers $m_j = d_j - 1$ are usually referred to as the exponents of the group G.

We begin by proving Steinberg's result, needed for the proof of Coleman's theorem. We require a preliminary lemma and employ the following terminology. Let $A = (a_{ij})$ be an $n \times n$ matrix with non-negative entries. We associate with A a graph $\mathcal{G}$ consisting of n nodes, connecting the nodes i, j iff $a_{ij} > 0$. A is said to be connected iff $\mathcal{G}$ is connected.

LEMMA 3.2. Let $A = (a_{ij})$ be a symmetric connected matrix. The largest eigenvalue λ of A is positive and a corresponding eigenvector e can be chosen all of whose entries are positive.

REMARK. The above is a special case of a theorem of Frobenius concerning the eigenvalues of matrices with non-negative entries [13]. Indeed the symmetry of A is not required. This extraneous assumption permits for a somewhat simpler proof and suffices for our purposes.

Proof. Let $Q(x) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j$ be the quadratic form associated with (a_{ij}). Then $\lambda = \operatorname{Max}_{\|x\|=1} Q(x) > 0$, where $\|x\|^2 = \sum_{i=1}^{n} x_i^2$. Choose $v = (v_1, \dots, v_n)$, $\|v\| = 1$, so that $Q(v) = \lambda$ and let $e = (e_1, \dots, e_n)$, where $e_i = |v_i|$, $1 \leqslant i \leqslant n$. Then $e_i \geqslant 0$, $1 \leqslant i \leqslant n$, and $\|e\| = 1$. As all $a_{ij} \geqslant 0$ and $\|e\| = 1$, we have $\lambda = Q(v) \leqslant Q(e) \leqslant \lambda$, so that $Q(e) = \lambda$. The latter implies $Ae = \lambda e$. It remains to show that each $e_i > 0$. Choose $e_j > 0$. Because of the connectivity assumption, we may choose $i_1, \dots, j_r = j$ so that $a_{ij_1}, a_{j_1 j_2}, \dots, a_{j_{r-1}, j}$ are all > 0. The relation $\lambda e_{j_{r-1}} = \sum_{k=1}^{n} a_{j_{r-1}, k} e_k$ shows that $e_{j_{r-1}} > 0$. Repeating this reasoning r times, we conclude that each $e_i > 0$.

THEOREM 3.9 (Steinberg [20]). *Let* $h =$ *order of* $R_1 \dots R_n$, $r =$ *number of reflections in* G. *Then* $r = \dfrac{nh}{2}$.

Proof. We may label the walls of the fundamental region F so that $W_1 \dots W_s$ are mutually perpendicular, and $W_{s+1}, \dots, W_n$ are mutually perpendicular (I.e. if the nodes corresponding to $W_1, \dots, W_s$ are black and those corresponding to $W_{s+1}, \dots, W_n$ are white, then each black node is linked only to white nodes and conversely). Let $E_1 = W_{s+1} \cap \dots \cap W_n$, $E_2 = W_1 \cap \dots \cap W_s$. Thus in terms of the dual basis $\{r_i'\}$, E_1 is the linear span of $r_1', \dots, r_s'$ and E_2 the linear span of $r_{s+1}', \dots, r_n'$. Let $S = R_{s+1} \dots R_n$, $T = R_1, \dots, R_s$ and denote the orthogonal complement of E_i, $i = 1, 2$, by $E_i^\perp$. The restriction of S to E_1, denoted by S_{E_1}, is the identity $r_{s+1}, \dots, r_n$ form a basis for $E_1^\perp$. Since they are orthogonal to each other, $R_i r_j = 0$ for $i \neq j$, $s + 1 \leqslant i, j \leqslant n$, so that $S_{E_1^\perp} = -$ identity. Similarly T_{E_2} $=$ identity, $T_{E_2^\perp} = -$ identity. We require the following

LEMMA 3.3. Let G_0 be the $n \times n$ matrix $((r_i, r_j))$ and I the $n \times n$ identity matrix. $I - G_0$ is connected. Thus, by Lemma 3.2, $I - G_0$ has a biggest positive eigenvalue λ and a corresponding eigenvector e with positive entries. Let $\sigma = \sum_{i=1}^{s} e_i r_i'$, $\tau = \sum_{i=s+1}^{n} e_i r_i'$ [1]. The plane π, determined by σ and τ, has non-trivial intersection with $E_1^\perp$ and $E_2^\perp$. It follows that S_π (T_π) is a reflection of π in the line through σ (τ).

Proof. The entries of $I - G_0$ are $\geqslant 0$, as $(r_i, r_j) \leqslant 0$ whenever $i \neq j$. The irreducibility of G is equivalent to saying that $I - G_0$ is connected. Let

$$G_0 = \begin{pmatrix} I & A \\ A' & I \end{pmatrix}, \quad G_0^{-1} = \begin{pmatrix} B & C \\ C' & D \end{pmatrix},$$

where A, C are $s \times n - s$ matrices (we use I to denote the identity matrix for various degrees; here degree $I = s$). The relations $r_i = \sum_{j=1}^{n} (r_i, r_j)\, r_j'$, $r_i' = \sum_{i=1}^{n} (r_i', r_j')\, r_j$, $1 \leqslant i \leqslant n$, show that $G_0^{-1} = ((r_i', r_j'))$. Since $G_0^{-1} G_0$ $= I$, we have

$$BA + C = C' + DA' = 0 \tag{3.1}$$

Let e^1 be the vector consisting of the first s components of e, e^2 the vector

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[1]) Geometrically, the directions of σ, τ are those in E_1, E_2 which produce the smallest angle. To prove this, one solves this minimum problem by the method of multipliers. Lagrange's equations lead to (3.2.).

consisting of the last $n - s$ components of e. The equation $(I - G_0)\, e = \lambda e$ becomes

$$A\, e^2 + \lambda\, e^1 = A'\, e^1 + \lambda\, e^2 = 0\,. \tag{3.2}$$

(3.1), (3.2) imply

$$\lambda B\, e^1 - C\, e^2 = \lambda D\, e^2 - C'\, e^1 = 0\,. \tag{3.3}$$

Let $\sigma = \sum_{1=1}^{s} e_i\, r_i'$, $\tau = \sum_{i=s+1}^{n} e_i\, r_i'$. (3.3) may be rewritten as

$$\begin{aligned} r_i' \cdot (\lambda\sigma - \tau) &= 0, \quad 1 \leqslant i \leqslant s\,, \\ r_i' \cdot (\lambda\tau - \sigma) &= 0, \quad s+1 \leqslant i \leqslant n\,. \end{aligned} \tag{3.4}$$

The vectors $\lambda\,\sigma - \tau$, $\lambda\,\tau - \sigma$ are $\neq 0$ and in π. (3.4) states that $\lambda\,\sigma - \tau \in E_1^{\perp}$, $\lambda\,\tau - \sigma \in E_2^{\perp}$. Since $\sigma \in E_1$, $\sigma' = \lambda\,\sigma - \tau \in E_1^{\perp}$, we have $S(\sigma) = \sigma$, $S(\sigma') = -\sigma'$. I.e. S_π is a reflection in the line through σ. Similarly, T_π is a reflection in the line through τ.

We now return to the proof of Theorem 3.9. Let H be the subgroup generated by S, T. H_π is the group generated by S_π, T_π. Let

$$F_0 = \{v \mid v = x\,\sigma + y\,\tau, x, y > 0\} = F \cap \pi\,.$$

F_0 is a fundamental region for H_π. For let $\gamma \in H$, $\gamma_\pi \neq I$. Then $\gamma \neq I$ and we have $\gamma_\pi\, F \cap F = \gamma\, F \cap F \cap \pi = \Phi$. R_π is a rotation of π through twice the angle between σ and τ. We show that ord $R_\pi = h$. For let ord $R_\pi = k$. Since $R^h = I$, $R_\pi^h = I$, we have $k \leqslant h$. Choose $p \in F_0$. $R^k(p) = R_\pi^k(p) = p$ so that $R^k\, F \cap F \neq \Phi \Rightarrow R^k = I \Rightarrow h \leqslant k$. Thus $h = k$. It follows that F_0 is an angular wedge of angular width $\dfrac{2\pi}{h}$ and H_π is a dihedral group of order $2h$. The h transforms of σ are contained in precisely $(n-s)$ r.h.'s. The h transforms of τ are contained in precisely s r.h.'s. Every r.h. of G has a non-trivial intersection with π. Since each of the transforms of F_0 is contained in a chamber of G and each chamber is free of r.h.'s, these r.h.'s meet π only at the transforms of σ and τ. Counting the r.h.'s at the transforms of σ and τ, we obtain the count $h\,s + h\,(n-s) = h\,n$. Each r.h. is however counted twice, as it intersects π in a line and thus meets two of the σ and τ transforms. Hence $r = \dfrac{h\,n}{2}$.

As a by product of the above proof, we obtain the following result required to establish Theorem 3.8.

THEOREM 3.10. *$\zeta = e^{2\pi i/h}$ is an eigenvalue of R. Corresponding to ζ, we may choose an eigenvector v not lying in any r.h.* (Note: if v is complex, then v is said to lie in the r.h. π iff $L(v) = 0$, $L(x) = 0$ being the equation of π).

Proof. Assume first that the R_i's are labeled as in the proof of Theorem 3.9; i.e. the walls $W_1, \ldots, W_s$ are mutually perpendicular as are also $W_{s+1}, \ldots, W_n$. Let π be the plane of Lemma 3.3. We choose two orthonormal vectors v_1, v_2 in π such that v_1 is not contained in any r.h. of G and

$$(3.5)\qquad \begin{aligned} R(v_1) &= \cos\frac{2\pi}{h} v_1 + \sin\frac{2\pi}{h} v_2 \\ R(v_2) &= -\sin\frac{2\pi}{h} v_1 + \cos\frac{2\pi}{h} v_2 \end{aligned}$$

Let $v = v_1 - iv_2$. We conclude from (3.5) that $R(v) = e^{2i\pi/h} v$. Thus v is an eigenvector corresponding to the eigenvalue $\zeta = e^{2i\pi/h}$. v is not in any r.h. of G as v_1 is not in any r.h. of G.

For an arbitrary labeling of indices, choose a permutation $i_1, \ldots, i_n$ of $1, \ldots, n$ so that the above reasoning applies to $R' = R_{i_1} \ldots R_{i_n}$. By Theorem 3.7. $R = R_1 \ldots R_n = \sigma R' \sigma^{-1}$ for some $\sigma \in G$. Hence $R(\sigma v) = \zeta(\sigma v)$. Since the r.h.'s are permuted by σ, we conclude that σv is also not contained in any r.h. of G.

We also require

THEOREM 3.11. *1 is not an eigenvalue of R.*

REMARK. In Theorem 3.12 we obtain the characteristic equation of R, from which we may obtain Theorem 3.11. The following proof is shorter and avoids any explicit matrix representation for R.

Proof. Let π be the r.h. corresponding to the root r and σ the reflection in π. Then $v' = \sigma v$ becomes

$$(3.6)\qquad v' = v - 2(v,r)\, r$$

Suppose that $R_1 \ldots R_n v = v$, $\Leftrightarrow R_2 \ldots R_n v = R_1 v$. Repeated application of (3.6) shows that $R_2 \ldots R_n v = v + \lambda_2 r_2 + \ldots + \lambda_n r_n$, $\lambda_2, \ldots, \lambda_n$ being real numbers depending on v. Hence

$$(3.7)\qquad v + \lambda_2 r_2 + \ldots + \lambda_n r_n = v - 2(v, r_1)\, r_1$$

Since $r_1, \ldots, r_n$ are linearly independent we must have $(v, r_1) = 0$ $\Leftrightarrow R_1 v = v$, so that $R_2 \ldots R_n v = v$. Repeating the reasoning, we conclude $(v, r_i) = 0$, $1 \leqslant i \leqslant n$, $\Rightarrow v = 0$. Thus 1 is not an eigenvalue of $R_1 \dots R_n$.

We can now provide the

Proof of Theorem 3.8. Let $v_1, \dots, v_n$ be linearly independent eigenvectors of R with v_1 chosenas in Theorem 3.10; i.e. v_1 corresponds to the eigenvalue $\zeta = e^{2i\pi/h}$ and does not lie in any r.h. of G. Let $x_1, \dots, x_n$ be a coordinate system adapted to $v_1, \dots, v_n$. As $R^h = I$, all eigenvalues of R are h-th roots of I. By Theorem 3.11, 1 is not an eigenvalue of R. Hence the eigenvalues of R are $\zeta^{m_1}, \dots, \zeta^{m_n}$ where $m_1 = 1$ and $1 \leqslant m_1 \leqslant \dots \leqslant m_n = h - 1$, $1 \leqslant i \leqslant n$. R is given by $x_i' = \zeta^{m_i} x_i$, $1 \leqslant i \leqslant n$.

Let $I_1, \dots, I_n$ be a basic set of homogeneous invariants of G of respective degrees $d_1 \leqslant \dots \leqslant d_n$. By Theorem 2.5,

$$J = \frac{\partial (I_1, \dots, I_n)}{\partial (x_1, \dots, x_n)} \neq 0$$

off the r.h.'s of G. Hence $J \neq 0$ whenever $x = (x_1, 0, \dots, 0)$, $x_1 \neq 0$. It follows that there exists a permutation $j = j(i)$ of 1 to n such that

$$\frac{\partial I_i}{\partial x_j}(x_1, 0, \dots, 0) \neq 0$$

for $x_1 \neq 0$ and $1 \leqslant i \leqslant n$. This means that the $x_1^{d_i - 1}$ coefficient of

$$\frac{\partial I_i}{\partial x_j} \neq 0 \Rightarrow x_1^{d_i - 1} x_j$$

coefficient of $I_i \neq 0$, $1 \leqslant i \leqslant n$. Hence each $x_1^{d_i - 1} x_j$ is invariant under R. I.e.

(3.8) $$(d_i - 1) + m_j \equiv 0 \pmod h, \ 1 \leqslant i \leqslant n$$

Rewrite (3.8) as

(3.9) $$d_i - 1 = (h - m_j) + \varepsilon_i h, \ 1 \leqslant i \leqslant n$$

where each ε_i is an integer $\geqslant 0$. Let $m_j' = h - m_j$. The eigenvalues of R occur in pairs, so that the set of numbers $\{m_j'\}$ is identical with $\{m_j\}$. Summing both sides of (3.9) from $i = 1$ to $i = n$, we get

(3.10) $$\sum_{i=1}^{n} (d_i - 1) = \sum_{j=1}^{n} m_j' + \left(\sum_{i=1}^{n} \varepsilon_i\right) h$$

By Theorem 2.2, $\sum_{i=1}^{n} (d_i - 1) = r$. Since

$$\sum_{j=1}^{n} m_j' = \sum_{j=1}^{n} (h - m_j) = n\,h - \sum_{j=1}^{n} m_j', \tag{3.11}$$

we also have $\sum_{j=1}^{n} m_j' = \dfrac{n\,h}{2}.$ We conclude from Theorem 3.9 that $\sum_{i=1}^{n} (d_i - 1) = \sum_{j=1}^{n} m_j'$. (3.10) shows that $\sum_{i=1}^{n} \varepsilon_i = 0 \Rightarrow \varepsilon_i = 0, 1 \leqslant i \leqslant n$. It follows from (3.9) that $d_i - 1 = m_i$, $1 \leqslant i \leqslant n$.

To make effective use of Coleman's Theorem, we need the explicit expression for the characteristic equation of R.

THEOREM 3.12 (Coxeter [5], p. 218). *The characteristic equation of* $R = R_1 \ldots R_n$ *is given by*

$$\begin{vmatrix} \dfrac{1+\lambda}{2} & \lambda\, a_{12} & \ldots & \lambda\, a_{1n} \\ a_{21} & \dfrac{1+\lambda}{2} & \lambda\, a_{23} \ldots & \lambda\, a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & \ldots & a_{n,n-1} & \dfrac{1+\lambda}{2} \end{vmatrix} = 0 \tag{3.12}$$

where $a_{ij} = -\cos(\pi/p_{ij})$, $1 \leqslant i, j \leqslant n$.

Proof. Let $v = \sigma\, v'$ where σ is a reflection in the r.h. perpendicular to the root r.
Then

$$v = v' - 2(v' \cdot r)\, r \tag{3.13}$$

We use (3.13) to obtain the matrix for R_j relative to the basis $r_1', \ldots, r_n'$. Let $v = \sum_{i=1}^{n} x_i\, r_i$, $v' = \sum_{i=1}^{r} x_i'\, r_i'$. Then $v' \cdot r_j = x_j', r_j = \sum_{i=1}^{n} a_{ij}\, r_i'$.
Substituting into (3.13), we get

$$v = R_j v' \Leftrightarrow x_i = x_i' - 2a_{ij}\, x'_j,\ 1 \leqslant i \leqslant n \tag{3.14}$$

Let

$$v = R_1\, v^{(1)},\ v^{(1)} = R_2\, v^{(2)},\ \ldots,\ v^{(n-1)} = R_n\, v^{(n)}$$

so that $v = R_1 \dots R_n v^{(n)}$. Suppose that $v^{(j)} = \sum_{i=1}^{n} x_i^{(j)} r_i', 1 \leqslant j \leqslant n$.
We conclude from (3.14) that

$$
(3.15) \quad \begin{cases} x_i = x_i' - 2a_{i1} x_1' \\ x_i' = x_i'' - 2a_{i2} x_2'' \\ \dots\dots\dots\dots\dots, 1 \leqslant i \leqslant n \\ x_i^{(n-1)} = x_i^{(n)} - 2a_{in} x_n^{(n)} \end{cases}
$$

Let $y_i = x^{(k)}$, $1 \leqslant i \leqslant n$. For each i we rewrite (3.15) as

$$
(3.16) \begin{cases} x_i' - x_i = 2a_{i1} y_1 \\ x_i'' - x_i' = 2a_{i2} y_2 \\ \dots\dots\dots\dots \\ y_i - x_i^{(i-1)} = 2a_{ii} y_i \end{cases} \qquad (3.17) \begin{cases} x_i^{(i+1)} - y_i = 2a_{i,i+1} y_{i+1} \\ x_i^{(i+2)} - x_i^{(i+1)} = 2a_{i,i+2} y_{i+2} \\ \dots\dots\dots\dots\dots\dots \\ x_i^{(n)} - x_i^{(n-1)} = 2a_{in} y_n \end{cases}
$$

Adding up respectively the equations in (3.16), and (3.17), we obtain

$$
(3.18) \qquad -x_i = \sum_{j=1}^{i-1} 2a_{ij} y_j + y_i, \ 1 \leqslant i \leqslant n
$$

$$
(3.19) \qquad x_i^{(n)} = \sum_{j=i+1}^{n} 2a_{ij} y_j + y_i, \ 1 \leqslant i \leqslant n
$$

(3.18), (3.19) may be abbreviated as

$$
(3.20) \qquad -x = Ay, x^{(n)} = A' y
$$

where

$$
(3.21) \quad A = \begin{bmatrix} 1 & & & & \\ 2a_{21} & & & & \\ . & 1 & & & \\ . & . & . & & \\ . & . & . & . & \\ 2a_{n1} & . & . & 2a_{n,\,n-1} & 1 \end{bmatrix}
$$

the entries above the diagonal being zero.

Hence $x = -A\,(A')^{-1}\,x^{(n)}$, so that $-A\,(A')^{-1}$ is the matrix for $R = R_1 \dots R_n$ relative to the basis $r_1', \dots, r_n'$. The characteristic equation for R is thus given by

$$|-A(A')^{-1} - \lambda I| = 0 \Leftrightarrow \left| \frac{A + \lambda A'}{2} \right| = 0 \tag{3.22}$$

which is the same as (3.12).

We rewrite the characteristic equation in a more symmetric form. Suppose first that G is of type I. We label nodes of the graphs in diagram 3.2 from left to right as $1, \ldots, n$. Thus $a_{ij} = 0$ whenever $|j - i| > 1$. Multiplying first the i-th row of the determinant in (3.12) by $\lambda^{(i-1)/2}$, $1 \leqslant i \leqslant n$, then the j-th column by $\lambda^{-j/2}$, $1 \leqslant j \leqslant n$, we get

$$\begin{vmatrix} \Lambda & & & & \\ & \cdot & & a_{ij} & \\ & & \cdot & & \\ & & & \cdot & \\ a_{ij} & & & \cdot & \\ & & & & \Lambda \end{vmatrix} = 0 \tag{3.23}$$

where $\Lambda = \dfrac{\lambda^{1/2} + \lambda^{-1/2}}{2}$

If G is of type II, then the nodes on the principal chain are labeled from left to right as 1 to $n - 1$, the remaining node being labeled n. The n^{th} node is linked to the q^{th} node. Let $i' = i, j' = j$, $1 \leqslant i, j \leqslant n - 1$, and $i' = j' = q + 1$ whenever i or $j = n$. Multiply first the i-th row of the determinant in (3.12) by $\lambda^{\frac{i'-1}{2}}$, $1 \leqslant i \leqslant n$, then the j-th column by $\lambda^{-j'/2}$. We obtain again (3.23). We have proven

COROLLARY. *The characteristic equation of* R *is given by* (3.23).

We illustrate the use of Coleman's Theorem by computing the d_i's for the icosahedral group I_3. In this case the characteristic equation (3.23) becomes

$$\begin{vmatrix} \Lambda & -\frac{1}{2} & 0 \\ -\frac{1}{2} & \Lambda & -\cos\frac{\pi}{5} \\ 0 & -\cos\frac{\pi}{5} & \Lambda \end{vmatrix} = 0 \tag{3.24}$$

The roots of (3.24) are readily computed to be $\zeta = e^{\frac{2\pi i}{10}}, \zeta^5, \zeta^9$. It follows from Coleman's Theorem that $d_1 = 2$, $d_2 = 6$, $d_3 = 10$.