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Artikel: CARTIER DUALITY AND FORMAL GROUPS OVER Z

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Kapitel: §2. Groups

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(i) *The contravariant functor $\mathscr{A} \to \mathscr{B}$ given by $A \mapsto A^*$ is an antiequivalence of categories taking products to coproducts and final objects to initial objects.*

(ii) *The restriction of this functor is an equivalence $(C\mathscr{A})^{op} \to G\mathscr{B}$.*

Several remarks are in order. First, we shall not define "linearly compact"; its role is to guarantee that A and A^{**} are isomorphic vector spaces, and this is false for discrete infinite dimensional spaces. Second, the proof of (ii) is a routine inspection of the various diagrams, once statement (i) has been proved.

There are at least two papers giving a Cartier duality between certain categories of commutative topological k-algebras and of cocommutative k-coalgebras, where k is a commutative ring. (Ditters [2]; Morris and Pareigis [5]). We present a version of Cartier duality between certain commutative Z-algebras (= commutative rings) and cocommutative Z-coalgebras; actually, our proof works if one replaces Z by any principal ideal domain that is neither a field nor a complete discrete valuation ring. Thus, our theorem is weaker than those of Ditters and Morris-Pareigis in that the ground rings k are restricted; it is stronger than their results in that we need not assume the algebras are topological algebras. Indeed, it is easy to see our category of commutative algebras is a proper, full subcategory of the corresponding categories of Ditters and of Morris-Pareigis. We add that our proof is quite easy and all details are given.

§2. Groups

All groups are abelian and are written additively.

Definition. A subgroup A' of a group A is *cofinite* if A/A' if f.g. free (f.g. abbreviates "finitely generated").

Of course, A' cofinite implies $A = A' \oplus A''$, where $A'' \cong A/A'$.

Definition. The *cofinite topology* on a group A is that (linear) topology having a fundamental system of neighborhoods of 0 consisting of all cofinite subgroups of A.

It is clear that A is a topological group in the cofinite topology.

Suppose $A = Z^I$ for some index set I. We may also topologize A with the *product topology*, i.e., equip each factor Z with the discrete topology and consider A in the corresponding product topology. The first lemma shows that the cofinite topology gives a coordinate-free description of the product topology.

LEMMA 1. *If* $A = Z^I$ *and* I *is countable, then the cofinite topology coincides with the product topology.*

Proof. It is easy to see that, in either topology (and for any index sets I and J), every homomorphism $f\colon Z^I \to Z^J$ is sequentially continuous (if $x_n \to x$, then $f(x_n) \to f(x)$); if we assume I and J countable, then Z^I and Z^J are first countable (even metrizable), and so f is continuous.

Assume A' is cofinite in A, and A has the product topology. For finite n, we see Z^n is discrete (in either topology), whence the natural map $\pi\colon A \to A/A' \cong Z^n$ is continuous and $A' = \pi^{-1}(\{0\})$ is open.

Now assume A has the cofinite topology. If $U_i = \prod_{j\in I} X_j$, where $X_j = Z$ if $j \neq i$ and $X_j = \{0\}$ if $j = i$, then U_i is cofinite, hence open. It follows easily that every basic open set in the product topology is open in cofinite topology.

One may prove that Lemma 1 is true for any set I whose cardinal is nonmeasurable [6].

DEFINITION. The *completion* of a group A is $\varprojlim A/A'$, where A' ranges over all cofinite subgroups of A; we denote $\varprojlim A/A'$ by $A^\wedge$. There is a canonical map $\lambda\colon A \to A^\wedge$; we say A is *complete* if λ is an isomorphism.

COROLLARY 2. *If* $A = Z^I$, *where* I *is countable, then* A *is complete.*

Proof: It is easy to see that, in the product topology, A is complete in the usual metric. By Lemma 1 and [4, Theorem 13.7], the two notions of completeness coincide.

The following remarkable result of Łos is the reason we need not mention linear compactness. Let us denote $\mathrm{Hom}_Z(A, Z)$ by A^*.

LEMMA 3. *(Łos)*

(i) *Let* $A = Z^N = \prod_{n=1}^{\infty} <e_n>$. *If* $G = Z$ *or* $G = Z^{(I)}$, *the direct sum of card* I *copies of* Z, *then the map* $f \mapsto (f|<e_n>)$ *is an isomorphism* $\mathrm{Hom}_Z(A, G) \xrightarrow{\sim} \sum_{n=1}^{\infty} \mathrm{Hom}_Z(<e_n>, G)$.

(ii) *If* I *is countable, then* $(Z^I)^* \cong Z^{(I)}$.

(iii) *If* I *is countable and either* $A = Z^I$ *or* $A = Z^{(I)}$, *then* A *is reflexive in the sense that the natural map* $A \to A^{**}$ *is an isomorphism.*

Proof: [4; §94]. This Lemma is true if Z is replaced by any principal ideal domain that is neither a field nor a complete discrete valuation ring.

Again the countability assumption is too strong; one only needs the cardinal of I nonmeasurable. Also, part (i) is true for groups G other than Z and $Z^{(I)}$, namely, "slender" groups.

For any index sets I and J, there is a natural imbedding $Z^I \otimes Z^J \to Z^{I \times J}$ given by $(m_i) \otimes (n_j) \mapsto (m_i \otimes n_j)$.

Lemma 4. *Assume I and J are countable. Then if $Z^I \otimes Z^J$ and $Z^{I \times J}$ are given the cofinite topology, then $Z^I \otimes Z^J$ is a dense subspace of $Z^{I \times J}$.*

Proof: By "subspace" we mean that the cofinite topology on $Z^I \otimes Z^J$ coincides with the relative topology $Z^I \otimes Z^J$ inherits from the larger space $Z^{I \times J}$. Let us write $A = Z^I \otimes Z^J$ and $G = Z^{I \times J}$. If G' is cofinite in G, then

$$A/G' \cap A \cong (A+G')/G' \subset G/G' ,$$

whence $G' \cap A$ is cofinite in A. Assume that A' is cofinite in A. Now A' is cofinite in A if and only if there are finitely many $f_i \in A^*$ with $A' = \cap \ker f_i$. Moreover, if $f \in A^*$ and $A' = \ker f$, then there exists a cofinite G' in G with $G' \cap A = A'$ if and only if there is $\tilde{f} \in G^*$ extending f. Thus it suffices to show we may extend $f \in (Z^I \otimes Z^J)^*$ to $\tilde{f} \in (Z^{I \times J})^*$. But this follows easily from the adjoint isomorphism and Lemma 3:

$$\begin{aligned} \mathrm{Hom}(Z^I \otimes Z^J, Z) &= \mathrm{Hom}(Z^I, \mathrm{Hom}(Z^J, Z)) \\ &= \mathrm{Hom}(Z^I, Z^{(J)}) \\ &= Z^{(I \times J)} = \mathrm{Hom}(Z^{I \times J}, Z) . \end{aligned}$$

We have shown that $Z^I \otimes Z^J$ is a subspace of $Z^{I \times J}$; it is dense because it contains the dense subgroup $Z^{(I \times J)}$.

We remark that Lemma 4 is false for some subgroups of $Z^{I \times J}$; for example, if $A = Z^{(I \times J)} \oplus \langle x \rangle$, where x has each coordinate 1, then $Z^{(I \times J)}$ is cofinite in A; the corresponding functional f on A cannot extend to $Z^{I \times J}$, for every $\tilde{f} \in (Z^{I \times J})^*$ that vanishes on $Z^{(I \times J)}$ must be 0 [4; Theorem 94.4].

Lemma 5. *If I and J are countable, there is a natural isomorphism*

$$(Z^I \otimes Z^J)^\wedge \cong (Z^{(I)} \otimes Z^{(J)})^* .$$

(Recall: $^\wedge$ means completion and $*$ means dual space).

Proof: Since $Z^{(I)} \otimes Z^{(J)} \cong Z^{(I\times J)}$, the right hand side is $Z^{I\times J}$. By Lemma 4, $Z^I \otimes Z^J$ is a dense subspace of $Z^{I\times J}$, so that both have the same completion. This finishes the argument, for $Z^{I\times J}$ is complete, by Corollary 2.

COROLLARY 6. *If I and J are countable, then $(Z^I \otimes Z^J)^\wedge \cong Z^K$, where K is countable.*

Proof: Indeed, we have just seen that we may take $K = I \times J$.

LEMMA 7. *Assume A and B torsion-free. If A' is cofinite in A and B' is cofinite in B, then there is a natural isomorphism*

$$A \otimes B/(A' \otimes B + A \otimes B') \cong A/A' \otimes B/B' .$$

Proof: Define $\theta: A \otimes B \to A/A' \otimes B/B'$ by $a \otimes b \mapsto \bar{a} \otimes \bar{b}$ (where bar denotes appropriate coset); let $K = \ker \theta$. As A and B are torsion-free, they are Z-flat, and so there is a commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r] & A \otimes B \arrow[r, "\theta"] & A/A' \otimes B/B' \arrow[r] & 0 \\
0 \arrow[r] & A' \otimes B + A \otimes B' \arrow[r] \arrow[u] & A \otimes B \arrow[r] \arrow[u, "="'] & A \otimes B/(A' \otimes B + A \otimes B') \arrow[r] \arrow[u, dashed] & 0
\end{tikzcd}$$

The dotted arrow exists and is an epimorphism, by diagram-chasing; it is an isomorphism because both right hand terms are f.g. free of the same rank (to compute the bottom quotient, observe that $A = A' \oplus A''$, $B = B' \oplus B''$, where $A'' \cong A/A'$ and $B'' \cong B/B'$).

LEMMA 8. *Let $A = Z^I$ and $B = Z^J$, where I and J are countable. The subgroups of $A \otimes B$ of the form $A' \otimes B + A \otimes B'$, where A' is cofinite in A and B' is cofinite in B, form a fundamental system of neighborhoods at 0 for the cofinite topology of $A \otimes B$.*

Proof: First of all, Lemma 7 shows that each of these special subgroups of $A \otimes B$ is cofinite.

Next, assume C is cofinite in $A \otimes B$, so there is an exact sequence

$$0 \longrightarrow C \longrightarrow A \otimes B \xrightarrow{\theta} F \longrightarrow 0$$

with F f.g. free. Define $A' = \{a \in A: \theta(a \otimes b) = 0 \text{ for all } b \in B\}$ and, similarly, $B' = \{b \in B: \theta(a \otimes b) = 0 \text{ for all } a \in A\}$. Clearly $A' \otimes B + A \otimes B' \subset C$. Now A' is pure in A and B' is pure in B, so that A/A' and B/B' are torsion-free. Also, A' is closed in A (and B' is closed in B)

because θ is continuous (I and J are countable), so that A/A' is complete. By considering maximal independent subsets of A and B and observing that only finitely many elements of A are involved in lifting a (finite) basis of F, we see that A/A' has finite rank (similarly for B/B'). As the only finite rank complete groups are f.g. free, it follows that A' and B' are cofinite.

§3. Formal Groups

Definition. Let $\mathscr{A}$ denote the category of all commutative rings with 1 whose underlying additive group is of the form Z^I, where card $I \leqslant \aleph_0$.

Note that $Z[[x_1, ..., x_n]]$, formal power series over Z in n variables, is an object of $\mathscr{A}$. Further, $\mathscr{A}$ has an initial object, namely, Z.

Lemma 9. *Every* $A \in \operatorname{obj} \mathscr{A}$ *is a complete topological ring in the cofinite topology.*

Proof: By Lemma 1 and Corollary 2, we know A is a complete topological group. It remains to show that multiplication $m: A \times A \to A$ is continuous, and, for this it suffices to prove the corresponding homomorphism $m': A \otimes A \to A$ is continuous; this is so because every homomorphism is continuous in the cofinite topology.

The next lemma is taken almost verbatim from [1; p. 12].

Lemma 10. *If* $A \in \operatorname{obj} \mathscr{A}$, *then* A *has a fundamental system of neighborhoods of* 0 *consisting of cofinite ideals.*

Proof: Let A' be a cofinite subgroup of A. Since multiplication is continuous, there is a cofinite subgroup W of A with $W^2 \subset A'$. Since W is cofinite, it has a f.g. free complement $\langle a_1, ..., a_r \rangle$. For each j, the continuity of $x \mapsto a_j \cdot x$ at 0 implies the existence of a cofinite $W_j \subset W$ with $a_j W_j \subset A'$. If $U = \bigcap_{j=1}^{r} W_j$, then U is cofinite in A. Moreover, $a_j U \subset A'$ for all j and $WU \subset A'$ (in fact, $W^2 \subset A'$ and $U \subset W$); hence $AU \subset A'$. Since $1 \in A$, we have $U \subset AU$, so that A/AU is f.g. Now if $(AU)_*$ is the pure subgroup of A generated by AU, then $(AU)_*$ is also an ideal, is cofinite, and $(AU)_* \subset A'_* = A'$ (for A' is already pure).

Lemma 11. *$\mathscr{A}$ has coproducts.*