5. SCHWARZIAN DERIVATIVE AND UNIVALENCE

$> \sigma_1 - \varepsilon$. If w is defined by $w(z) = z + e^{i\theta} / z$ and $f = w \circ h^{-1}$, then f is univalent in A and

$$\| S_f \|_A = \| S_w - S_h \|_D \geqslant | S_w(0) - S_h(0) | = | 6e^{i\theta} + S_h(0) | .$$

By choosing φ suitably we obtain $\| S_f \|_A > 6 + \sigma_1 - \varepsilon$.

5. Schwarzian derivative and univalence

5.1 *Constant* σ_3. Let A again be a simply connected domain with more than one boundary point. As a kind of opposite to the constant σ_2 we define

$$\sigma_3 = \sup \{a \mid \| S_f \| \leqslant a \text{ implies } f \text{ univalent in } A\}.$$

Note that the number $a = 0$ is always in the above set. In this definition, sup can be replaced by max, as can be shown by a standard normal family argument.

Nehari [12] proved that in a disc, the condition $\| S_f \| \leqslant 2$ implies the univalence of f, and Hille [5] showed that the bound 2 is best possible. In other words, $\sigma_3 = 2$ for a disc.

A closer study of σ_3 leads to the universal Teichmüller space and reveals an intrinsic significance of quasiconformal mappings in the theory of univalent functions. The gist is the following result.

THEOREM 5.1. *The constant* σ_3 *is positive if and only if* A *is bounded by a quasicircle.*

Proof: The sufficiency of the condition was established by Ahlfors [1] who actually proved more: If A is bounded by a K-quasicircle, there is an $\varepsilon > 0$ depending only on K, such that whenever $\| S_f \|_A < \varepsilon$, then f is univalent and can be continued to a quasiconformal mapping of the plane. In the proof, the extension of the given meromorphic f is explicitly constructed by means of a continuously differentiable quasiconformal reflection φ in ∂A with bounded $| d\varphi | / | dz |$ (cf. 3.3).

The necessity was proved by Gehring [2]. His proof was in two steps. It was first shown, by aid of an example, that if A is not b-locally connected for any b, then $\sigma_3 = 0$. After this, the desired conclusion was drawn from the result we stated above as Lemma 3.2.

5.2 *Universal Teichmüller space.* Henceforth, we assume that the domain A is bounded by a quasicircle. Let $Q(A)$ be the Banach space consisting of all holomorphic functions φ of A with finite norm. We introduce the subsets

$$U(A) = \{\varphi = S_f \mid f \text{ univalent in } A\},$$
$$T(A) = \{S_f \in U(A) \mid f \text{ can be extended to a quasiconformal mapping of the plane}\}.$$

Both sets are well defined. The set $T(A)$ is called the *universal Teichmüller space* of A.

THEOREM 5.2. *The sets* $T(A)$ *and* $U(A)$ *are connected by the relation* $T(A)$ = interior of $U(A)$.

Proof: We first show that $T(A)$ is open. Choose $S_f \in T(A)$, $S_h \in Q(A)$, and set $g = h \circ f^{-1}$. Then g is meromorphic in the domain $f(A)$. Since ∂A is a quasicircle, $\partial f(A)$ is also a quasicircle. By the theorem of Ahlfors cited in the proof of Theorem 5.1, there is an $\varepsilon > 0$ such that if

$$\| S_g \|_{f(A)} < \varepsilon, \tag{5.1}$$

then $S_g \in T(f(A))$. Now, choose h so that $\| S_f - S_h \|_A < \varepsilon$. Then (5.1) holds, and it follows that $S_h = S_{g \circ f} \in T(A)$.

After this it suffices to prove that int $U(A) \subset T(A)$. Choose $S_f \in$ int $U(A)$ and then an $\varepsilon > 0$, so that the ball $B = \{\varphi \in Q(A) \mid \| \varphi - S_f \| < \varepsilon\}$ is contained in $U(A)$. Let g be an arbitrary meromorphic function in $f(A)$ for which $\| S_g \|_{f(A)} < \varepsilon$. If $h = g \circ f$, then $\| S_f - S_h \|_A = \| S_g \|_{f(A)} < \varepsilon$. Thus $S_h \in U(A)$. But then also $g = h \circ f^{-1}$ is univalent, and we have proved that σ_3 is positive for the domain $f(A)$. By Theorem 5.1, the boundary $\partial f(A)$ is a quasicircle. Hence, by the remark in 3.3, $S_f \in T(A)$.

COROLLARY 5.1. *If* f *is univalent in* A *and* $\| S_f \|_A < \sigma_3$, *then* f *can be extended to a quasiconformal mapping of the plane.*

Proof: This follows immediately from Theorem 5.2, in view of our previous remark that the closed ball $\{\varphi \in Q(A) \mid \| \varphi \|_A \leqslant \sigma_3\}$ is contained in $U(A)$.

By this Corollary, we have for A,

$$\sigma_3 = \sup \{a \mid \| S_f \|_A < a \text{ implies that } f \text{ is univalent and can be extended to a quasiconformal mapping of the plane}\}.$$

5.3 *New characterization for* σ_3. Theorem 5.2 was proved by Gehring [2] in the case where A is a half-plane. As is seen from the above proof, the generalization for an arbitrary A is immediate. In fact, the sets $Q(A)$, $U(A)$ and $T(A)$ corresponding to different domains A are isomorphic:

LEMMA 5.1. *Let h be a conformal mapping of the upper half-plane H onto A. Then the mapping $h^*: Q(A) \to Q(H)$, defined by $h^*(S_f) = S_{f \circ h}$, is a bijective isometry. It maps $U(A)$ and $T(A)$ onto $U(H)$ and $T(H)$, respectively.*

Proof: Clearly h^* is well defined and a bijection of $Q(A)$, $U(A)$ and $T(A)$ onto $Q(H)$, $U(H)$ and $T(H)$, respectively. That h^* is an isometry follows from formula (4.3).

The function h^* maps the origin of $Q(A)$ onto the point $S_h \in T(H)$, which has the distance σ_1 from the origin of $Q(H)$. If $B = \{\varphi \in Q(A) \mid \|\varphi\|_A \leqslant \sigma_3\}$, then

$$h^*(B) = \{\psi \in Q(H) \mid \|\psi - S_h\| \leqslant \sigma_3\}.$$

From this and the definition of σ_3 we infer that σ_3 is equal to the distance from the point S_h to the boundary of $U(H)$. The following characterization seems to be more useful:

LEMMA 5.2. *The constant σ_3 of A is equal to the distance of the point S_h to the boundary of $T(H)$.*

Proof: Let d denote the distance function in Q. Since $T(H) \subset U(H)$ we conclude from what we just said above that $\sigma_3 \geqslant d(\{S_h\}, U(H) - T(H))$. On the other hand, it follows from Theorem 5.2 that int $B \subset T(A)$ and hence int $h^*(B) \subset T(H)$. Therefore, $\sigma_3 \leqslant d(\{S_h\}, U(H) - T(H))$.

A standard normal family argument shows that $U(A)$ is a closed subset of $Q(A)$. Therefore, the closure of $T(A)$ is contained in $U(A)$. Gehring [3] showed recently that this inclusion is proper, thus disproving a famous conjecture of Bers.

However, it is true that on every sphere $\|\varphi\| = r$ of $Q(H)$, $2 \leqslant r \leqslant 6$, there are points of $U(H) - T(H)$ which belong to the closure of $T(H)$ ([9]).

5.4 *Estimates for* σ_3. Lemma 5.2 can be used to deriving estimates for σ_3 in terms of σ_1 ([9]). Suppose first that $0 \leqslant \sigma_1 < 2$. Then S_h lies in the ball $\{\varphi \in Q(H) \mid \|\varphi\| < 2\}$ which is a subset of $T(H)$. Since $\|S_h\| = \sigma_1$, we conclude that $d(\{S_h\}, U(H) - T(H)) \geqslant 2 - \sigma_1$. Consequently, by Lemma 5.2,

$$\sigma_3 \geqslant 2 - \sigma_1 \, . \tag{5.2}$$

In order to prove that this inequality is sharp, we consider the point S_w, where w is the restriction to H of a branch of the logarithm. Since the boundary of $w(H)$ is not a quasicircle, $S_w \in U(H) - T(H)$. From $S_w(z) = z^{-2}/2$ it follows that $\| S_w \|_H = 2$. Let h be determined by the condition $S_h = r\, S_w$, $0 < r < 1$, and set $A = h(H)$. From $\| S_h \|_H < 2$ it follows that $S_h \in T(H)$, and so ∂A is a quasicircle. Now

$$\sigma_3 = d(\{S_h\}, U(H) - T(H)) = \| S_w - S_h \| = 2(1-r) = 2 - \sigma_1 \, ,$$

showing that (5.2) is sharp.

Suppose that $2 \leqslant \sigma_1 < 6$. We then conclude from the remark at the end of 5.3 that, even though $\sigma_3 > 0$ for each A, we have $\inf \sigma_3 = 0$ for every σ_1.

Similarly, Lemma 5.2 can be used to deriving the upper estimate

$$\sigma_3 \leqslant \min\,(2, 6 - \sigma_1)\, .$$

(For the details we refer to [9].)

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