

# INTEGRAL REPRESENTATION THEOREMS VIA BANACH ALGEBRAS

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Objekttyp: **Article**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **25 (1979)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **25.05.2024**

Persistenter Link: <https://doi.org/10.5169/seals-50383>

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# INTEGRAL REPRESENTATION THEOREMS VIA BANACH ALGEBRAS

by George MALTESE

## 1. INTRODUCTION

Many classical integral representation theorems of analysis can be obtained as special cases of the Choquet Representation Theorem [6], [7], [14] or the Krein-Milman Theorem. The procedure involves the definition of a suitable convex compact set in some locally convex space and an explicit description of the extreme points of this set. The latter is often a non-trivial task, therefore it seems appropriate to develop alternative methods which are general enough to yield a class of integral representation theorems. In many situations in which an integral representation formula is sought, there is a natural commutative Banach algebra inherent in the background. For example in the case of Bochner's theorem for positive definite functions on a locally compact abelian group  $G$ , the natural Banach algebra is the convolution algebra  $L^1(G)$ . In the case of the Schoenberg-Eberlein theorem for Fourier-Stieltjes transforms on locally compact abelian groups, the Banach algebra is again the convolution algebra. In the case of the Spectral Theorem for a normal operator  $T$  on a Hilbert space  $\mathcal{H}$ , the natural Banach algebra is the closed commutative  $*$  algebra generated by  $T$  and the identity operator.

In this paper we show that the above mentioned theorems are all special cases of a general result (Theorem 1) on the integral representation of certain linear forms defined on commutative Banach algebras. Specialization of Theorem 1 to symmetric Banach algebras yields a generalized version (Theorem 2) of a result of Raikov [10] for positive functionals on such algebras.

The proof of Theorem 1 is straight forward and its version for positive functionals on involution algebras is classical [11]. The main point here is the relative ease of application of Theorem 1 to a variety of situations.

## 2. INTEGRAL REPRESENTATION THEOREMS FOR LINEAR FUNCTIONALS

Let  $A$  be a commutative Banach algebra over  $\mathbb{C}$  and let  $\Delta$  denote the locally compact space of regular maximal ideals of  $A$ . For each  $x \in A$  we use  $\hat{x}$  to denote the Gelfand-transform; i.e.,  $\hat{x}$  is the continuous mapping from  $\Delta$  to  $\mathbb{C}$  defined by the relations:

$$\hat{x}(m) = m(x) \quad \text{for } m \in \Delta.$$

By  $C_0(\Delta)$  we shall denote the algebra of all complex-valued continuous functions on  $\Delta$  which vanish at infinity. For any subset  $\mathcal{A} \subset A$  we shall use the notation  $\hat{\mathcal{A}}$  to denote the set  $\{\hat{x} : x \in \mathcal{A}\}$ . As usual  $\|\hat{x}\|_\infty$  denotes the supremum norm.

**THEOREM 1.** Let  $f$  be a linear form on the complex commutative Banach algebra  $A$  and let  $\mathcal{A}$  be a linear subspace of  $A$ . The following two statements are equivalent:

(1) There exists a constant  $M$  such that

$$|f(x)| \leq M \|\hat{x}\|_\infty \quad \text{for every } x \in \mathcal{A}.$$

(2) There exists a bounded complex Radon measure  $\mu$  on  $\Delta$  such that

$$f(x) = \int_{\Delta} \hat{x}(m) d\mu(m) \quad \text{for every } x \in \mathcal{A}.$$

*Proof.* The implication (2)  $\Rightarrow$  (1) is clear with  $M = \|\mu\|$ . We shall prove (1)  $\Rightarrow$  (2). Define a mapping  $L : \hat{\mathcal{A}} \rightarrow \mathbb{C}$  by

$$L(\hat{x}) = f(x).$$

It follows from (1) that  $L$  is well-defined, and that

$$|L(\hat{x})| \leq M \|\hat{x}\|_\infty \quad \text{for every } \hat{x} \in \hat{\mathcal{A}}$$

and so  $L$  is continuous with  $\|L\| \leq M$ . Using the Hahn-Banach Theorem we can extend  $L$  to a bounded linear form  $L_0$  on  $C_0(\Delta)$  and by the Riesz Representation Theorem we obtain the existence of a bounded complex Radon measure  $\mu$  on  $\Delta$  such that

$$\|\mu\| = \|L\| = \|L_0\| \quad \text{and}$$

$$L_0(\varphi) = \int_{\Delta} \varphi(m) d\mu(m) \quad \text{for every } \varphi \in C_0(\Delta).$$

In particular

$$f(x) = L(\hat{x}) = L_0(\hat{x}) = \int_{\Delta} \hat{x}(m) d\mu(m) \quad \text{for every } x \in \mathcal{A}.$$

*Remark:* Suppose that  $A$  has an identity and that  $\hat{A}$  is closed under complex conjugation, then since  $\hat{A}$  contains constants and separates the points of  $\Delta$ , the Stone-Weierstraß Theorem implies that  $\hat{A}$  is dense in  $C(\Delta)$ , the algebra of all complex-valued continuous functions on the compact Hausdorff space  $\Delta$ . If we impose these additional conditions on  $A$  and if we take  $\mathcal{A} = A$  in Theorem 1, we can conclude that in this case the representing measure  $\mu$  is uniquely determined.

If the algebra  $A$  has a continuous involution, one can use Theorem 1 to derive an extended version of a theorem due to Raikov [10]. We proceed to describe the situation.

Let  $A$  be a complex commutative Banach algebra with an isometric involution  $*$  and a bounded approximate identity  $\{u_\lambda\}_{\lambda \in \Lambda}$  i.e., a net satisfying the following conditions:

$$\begin{aligned} \|u_\lambda\| &\leq 1 \quad \text{for each } \lambda \in \Lambda, \\ \|u_\lambda x - x\| &\rightarrow 0 \quad \text{for each } x \in A. \end{aligned}$$

A continuous *positive* functional on  $A$  is an element  $f \in A'$  such that  $f(x^*x) \geq 0$  for every  $x \in A$ . If  $f$  is a continuous positive functional on  $A$  then the Cauchy-Schwarz inequality is valid (Dixmier [8, p. 23]) and this implies the following facts:

$$\begin{aligned} f(u_\lambda) &\rightarrow \|f\| \\ |f(x)|^2 &\leq \|f\| f(x^*x) \quad \text{for every } x \in A. \end{aligned}$$

If the involution is *symmetric*, which means  $(x^*)^\wedge = \overline{\hat{x}}$  for every  $x \in A$  or, equivalently, that every  $m \in \Delta$  is a *positive* linear functional, then by modifying a classical method of Gelfand-Raikov-Silov [10; p. 62] one can prove that

$$|f(x)| \leq \|f\| \|\hat{x}\|_\infty \quad \text{for every } x \in A.$$



As a corollary to Theorem 1 and the above discussion we obtain the following extended theorem of Raikov [10; p. 64], see also Bucy-Maltese [4]):

**THEOREM 2.** Let  $A$  be a commutative Banach algebra with an isometric involution which is symmetric. Suppose that  $A$  has a bounded approximate identity and let  $f \in A'$  be a continuous positive functional. Then there exists a unique positive Radon measure  $\mu$  on  $\Delta$  such that  $\|\mu\| = \|f\|$  and

$$f(x) = \int_{\Delta} \hat{x}(m) d\mu(m) \quad \text{for every } x \in A.$$

*Proof.* From the above remarks we know that

$$|f(x)| \leq \|f\| \|\hat{x}\|_{\infty} \quad \text{for every } x \in A.$$

By Theorem 1 there exists a complex Radon measure  $\mu$  on  $\Delta$  such that  $\|\mu\| \leq \|f\|$  and

$$f(x) = \int_{\Delta} \hat{x}(m) d\mu(m) \quad \text{for every } x \in A.$$

This formula implies

$$|f(x)| \leq \|\hat{x}\|_{\infty} \|\mu\| \leq \|x\| \|\mu\| \quad \text{for every } x \in A,$$

so that  $\|f\| \leq \|\mu\|$  and hence  $\|f\| = \|\mu\|$ .

Since  $\hat{A}$  is a self-adjoint subalgebra of  $C_0(\Delta)$  which separates points and for each  $m \in \Delta$  contains a function  $\hat{x}$  such that  $\hat{x}(m) \neq 0$  (in fact <sup>1)</sup> there exists an element  $u_{\beta}$  of the approximate identity such that  $\hat{u}_{\beta}(m) \neq 0$ ), the Stone-Weierstraß Theorem implies the uniqueness of the measure  $\mu$ . The positivity of  $\mu$  also follows from the fact that  $\hat{A}$  is dense in  $C_0(\Delta)$ . In fact if  $p$  is a non-negative function in  $C_0(\Delta)$ , then  $p = |q|^2$  for some  $q \in C_0(\Delta)$ . Choose a sequence  $\{x_n\}$  in  $A$  such that

$$\hat{x}_n \rightarrow q.$$

<sup>1)</sup> If  $m \in \Delta$ , then  $\|m\| \neq 0$  and by the assumption of symmetry  $m$  is a positive functional. Therefore, as mentioned above,  $\|m\| = \lim_{\alpha} m(u_{\alpha})$  so that there must exist some  $u_{\beta}$  of the approximate identity such that  $m(u_{\beta}) \neq 0$ .

Since  $(x_n^*)^\wedge = \overline{x_n}$  it follows that  $(x_n^*)^\wedge \rightarrow \bar{q}$  and hence

$$(x_n x_n^*)^\wedge \rightarrow |q|^2 = p.$$

This implies

$$\begin{aligned} \int_A p(m) d\mu(m) &= \lim_n \int_A (x_n^* x_n)^\wedge(m) d\mu(m) \\ &= \lim_n f(x_n^* x_n) \geq 0, \end{aligned}$$

so that  $\mu$  is a positive measure and this completes the proof.

If  $A$  has an identity, as is the case in Raikov's original version, the above proof can be somewhat simplified.

**THEOREM 3 (Raikov).** Let  $A$  be a complex commutative Banach algebra with an identity  $e$  and with an isometric involution which is symmetric. If  $f$  is a continuous positive functional on  $A$ , then there exists a unique positive Radon measure  $\mu$  on  $\Delta$  such that  $\|\mu\| = \|f\|$  and

$$f(x) = \int_A \hat{x}(m) d\mu(m) \quad \text{for every } x \in A.$$

*Proof.* As above we know that

$$|f(x)| \leq \|f\| \|\hat{x}\|_\infty \quad \text{for every } x \in A.$$

From Theorem 1 there exists a complex Radon measure  $\mu$  on  $\Delta$  such that  $\|\mu\| \leq \|f\|$  and

$$f(x) = \int_A \hat{x}(m) d\mu(m) \quad \text{for every } x \in A.$$

Hence  $\|\mu\| \leq \|f\| = f(e) = \mu(1) \leq \|\mu\|$  so that  $\mu(1) = \|\mu\|$  which is enough to imply that  $\mu$  is positive. The uniqueness of  $\mu$  follows as in the Remark to Theorem 1.

### 3. APPLICATIONS OF THE INTEGRAL REPRESENTATION THEOREMS

*Application 1 (Bochner's Theorem).* Let  $G$  be a locally compact abelian group and let  $\hat{G}$  denote the (locally compact) character group. Denote

Haar measure on  $G$  by  $dt$  and the group algebra (with convolution as multiplication) by  $L^1(G)$ .

*Definition* : A function  $p \in L^\infty(G)$  is said to be *positive definite* provided

$$\int_G \int_G F(t) \overline{F(s)} p(t-s) dt ds \geq 0$$

for every  $F \in L^1(G)$ .

Using the natural involution  $F \rightarrow \tilde{F}$  of  $L^1(G)$  defined by  $\tilde{F}(t) = \overline{F(-t)}$ , we can rewrite the definition of positive definiteness as follows:  $p \in L^\infty(G)$  is positive definite provided

$$\int_G F^* \tilde{F}(t) p(t) dt \geq 0 \quad \text{for all } F \in L^1(G).$$

For a positive definite function  $p$  define the mapping  $P : L^1(G) \rightarrow \mathbb{C}$  as follows:

$$P(F) = \int_G F(t) p(t) dt.$$

Therefore  $P$  is a continuous positive linear functional on the symmetric involution algebra  $L^1(G)$  such that  $\|P\| = \|p\|_\infty$ . By the discussion preceding Theorem 2 and the fact that  $L^1(G)$  has an approximate identity, we know that

$$|P(F)| \leq \|P\| \|\hat{F}\|_\infty \quad \text{for every } F \in L^1(G),$$

where  $\hat{F}$  is the Fourier-Gelfand transform. By Theorem 2 there exists a unique positive Radon measure  $\mu$  on  $\Delta(L^1(G))$  such that

$$P(F) = \int_{\Delta(L^1(G))} \hat{F}(\Gamma) d\mu(\Gamma) \quad \text{for every } F \in L^1(G).$$

It is classical that  $\Delta(L^1(G))$  is homeomorphic to the locally compact character group  $\hat{G}$  under the correspondence  $\Gamma \leftrightarrow \gamma$  where

$$\Gamma(F) = \hat{F}(\gamma) \quad \text{for every } F \in L^1(G).$$

Therefore using the same symbol for the measure induced on  $\hat{G}$  by  $\mu$  under this identification, we obtain

$$P(F) = \int_{\hat{G}} \hat{F}(\gamma) d\mu(\gamma) \quad \text{for every } F \in L^1(G)$$

which implies

$$\begin{aligned}\int_G F(t) p(t) dt &= \int_{\hat{G}} d\mu(\gamma) \int_G F(t) \overline{\gamma(t)} dt \\ &= \int_G F(t) \int_{\hat{G}} \overline{\gamma(t)} d\mu(\gamma) dt.\end{aligned}$$

Since this holds for all  $F \in L^1(G)$  we conclude that

$$p(t) = \int_{\hat{G}} \overline{\gamma(t)} d\mu(\gamma) \quad \text{for almost all } t.$$

This is, of course, the famous Bochner characterization of positive definite functions ([5], [6], [7], [10], [11], [16]).

*Application 2* (A theorem of Schoenberg-Eberlein). A complex-valued function  $\psi$  defined on the locally compact abelian group  $G$  is called a *Fourier-Stieltjes transform* if there exists a bounded Radon measure  $\nu_\psi$  defined on the dual group  $\hat{G}$  such that

$$\psi(t) = \int_{\hat{G}} \overline{\gamma(t)} d\nu_\psi(\gamma) \quad \text{for almost all } t \in G.$$

*Definition*: A measurable complex-valued function  $\psi$  on  $G$  satisfies the *Schoenberg condition* provided

- (a)  $\psi$  is integrable on every compact set;
- (b) there exists a constant  $M$  such that

$$\left| \int_G F(t) \psi(t) dt \right| \leq M \sup_{\gamma} \left| \int_G F(t) \overline{\gamma(t)} dt \right|$$

for every  $F \in K(G)$ , where  $K(G)$  is the set of continuous functions on  $G$  with compact support.

The following theorem is due to Schoenberg [17] for the case  $G = \mathbf{R}$  and to Eberlein [9] in the general case.

**THEOREM.** A measurable complex-valued  $\psi$  has a representation as a Fourier-Stieltjes transform if and only if  $\psi$  satisfies the Schoenberg condition.

*Proof.* It is immediate that if  $\psi$  is a Fourier-Stieltjes transform then  $\psi$  satisfies the Schoenberg condition for the constant  $M = \|\nu_\psi\|$ .

If now  $\psi$  satisfies the Schoenberg condition define  $L : L^1(G) \rightarrow \mathbf{C}$  as usual by

$$L(F) = \int_G F(t) \psi(t) dt.$$

By hypothesis

$$|L(F)| \leq M \|\hat{F}\|_{\infty} \quad \text{for every } F \in K(G).$$

From Theorem 1 there exists a bounded Radon measure  $\nu_{\psi}$  defined on  $\hat{G}$  such that

$$L(F) = \int_{\hat{G}} \hat{F}(\gamma) d\nu_{\psi}(\gamma) \quad \text{for } F \in K(G).$$

Therefore it follows that

$$\int_G F(t) \psi(t) dt = \int_{\hat{G}} d\nu_{\psi}(\gamma) \int_G F(t) \overline{\gamma(t)} dt \quad \text{for all } F \in K(G)$$

and hence

$$\psi(t) = \int_{\hat{G}} \overline{\gamma(t)} d\nu_{\psi}(\gamma) \quad \text{for almost all } t \in G,$$

which completes the proof.

*Application 3* (Positive definite functions on abelian semigroups). Let  $S$  be an abelian semigroup, i.e. a set equipped with a composition law denoted by  $+$  such that the commutative and associative laws are valid. We shall assume the existence of a neutral element  $0$ . By  $l^1(S)$  we shall denote the real commutative Banach algebra of functions  $f : S \rightarrow \mathbb{R}$  with the property that

$$\|f\| = \sum_{a \in S} |f(a)| < \infty.$$

Multiplication in  $l^1(S)$  is defined by convolution; viz.,

$$f * g(a) = \sum_{\substack{s, t \in S \\ s+t=a}} f(s) g(t).$$

By a *character* on  $S$  we shall mean a function  $\gamma : S \rightarrow [-1, 1]$  which satisfies

- (i)  $\gamma(0) = 1$
- (ii)  $\gamma(s+t) = \gamma(s) \gamma(t)$  for all  $s, t \in S$ .

The set  $\hat{S}$  of all characters is an abelian semigroup under pointwise multiplication. If we endow  $\hat{S}$  with the topology of pointwise convergence, then  $\hat{S}$  is a compact Hausdorff space. By  $\Delta(l^1(S))$  we shall denote the

compact subset of  $l^1(S)'$  consisting of all continuous homomorphisms from  $l^1(S)$  onto  $\mathbf{R}$  endowed with the weak \* topology.

If  $\gamma \in \hat{S}$  then the mapping

$$\Gamma_\gamma : l^1(S) \rightarrow \mathbf{R}$$

defined by the relations

$$\Gamma_\gamma(f) = \sum_{s \in S} f(s) \gamma(s) \quad \text{for } f \in l^1(S)$$

is a non-trivial continuous homomorphism from  $l^1(S)$  onto  $\mathbf{R}$ , so that  $\Gamma_\gamma \in \Delta(l^1(S))$ . Conversely for every  $\Gamma \in \Delta(l^1(S))$  there exists  $\gamma \in \hat{S}$  with

$$\Gamma(f) = \sum_{s \in S} f(s) \gamma(s) \quad \text{for } f \in l^1(S).$$

It is easily verified that the mapping

$$\gamma \rightarrow \Gamma_\gamma$$

is a homeomorphism of  $\hat{S}$  onto  $\Delta(l^1(S))$ . We shall identify  $\hat{S}$  with  $\Delta(l^1(S))$  via this homeomorphism and consider the Gelfand transform  $\hat{f}$  of  $f \in l^1(S)$  as a continuous function on  $\hat{S}$  via

$$\hat{f}(\gamma) = \Gamma_\gamma(f).$$

A bounded function  $\varphi : S \rightarrow \mathbf{R}$  is called *positive definite* if

$$\sum_{s \in S} f * f(s) \varphi(s) \geq 0 \quad \text{for all } f \in l^1(S).$$

In the sequel we shall *assume* that the spectral radius formula is valid in the real Banach algebra  $l^1(S)$ ; i.e., we shall assume that

$$\|\hat{f}\|_\infty = \lim_{n \rightarrow \infty} \|f^n\|^{\frac{1}{n}}$$

for every  $f \in l^1(S)$ . One can easily prove that the spectral radius formula is true for the simple functions  $\varepsilon_a$ ,  $\varepsilon_a + \varepsilon_b$ ,  $\lambda \varepsilon_a$  and  $\varepsilon_a * \varepsilon_b$ , where for each  $a \in S$  the function  $\varepsilon_a$  is defined by the relations

$$\varepsilon_a(s) = \begin{cases} 0 & \text{for } s \neq a \\ 1 & \text{for } s = a. \end{cases}$$

In the general context of real Banach algebras, of course, the spectral radius formula need not hold.

The following theorem was demonstrated *without* the assumption of the spectral radius formula for  $l^1(S)$ . The theorem was obtained by Berg-Christensen-Ressel [3] using the Krein-Milman Theorem along with an interesting characterization of the extreme points of the convex compact set of normalized positive definite functions.

**THEOREM.** Assume that the spectral radius formula is valid for each  $f \in l^1(S)$ . If  $\varphi$  is a positive definite function on  $S$ , then there exists a unique positive Radon measure  $\mu$  on  $\hat{S}$  such that

$$\varphi(s) = \int_{\hat{S}} \gamma(s) d\mu(\gamma) \quad \text{for every } s \in S.$$

*Proof.* The functional  $L : l^1(S) \rightarrow \mathbf{R}$  defined by

$$L(f) = \sum_{s \in S} f(s) \varphi(s)$$

is positive, i.e.  $L(f * f) \geq 0$ . By our assumption of the validity of the spectral radius formula we have

$$|L(f)| \leq \varphi(0) \|\hat{f}\|_{\infty} \quad \text{for every } f \in l^1(S).$$

Exactly as in Theorem 1 and Theorem 3 there exists a positive Radon measure  $\mu$  on  $\hat{S}$  such that

$$L(f) = \int_{\hat{S}} f(\gamma) d\mu(\gamma) \quad \text{for every } f \in l^1(S).$$

Hence

$$\begin{aligned} \sum_{s \in S} f(s) \varphi(s) &= \int_{\hat{S}} \left( \sum_{s \in S} f(s) \gamma(s) \right) d\mu(\gamma) \\ &= \sum_{s \in S} f(s) \int_{\hat{S}} \gamma(s) d\mu(\gamma) \quad \text{for every } f \in l^1(S) \end{aligned}$$

and therefore we conclude that

$$\varphi(s) = \int_{\hat{S}} \gamma(s) d\mu(\gamma) \quad \text{for every } s \in S.$$

The uniqueness of  $\mu$  is a consequence of the Stone-Weierstrass Theorem: the algebra  $\widehat{l^1(S)}$  is dense in the algebra of all real continuous functions on  $\hat{S}$ .

*Application 4* (The spectral theorem for normal operators). Let  $\mathcal{H}$  be a Hilbert space and let  $\mathcal{L}(\mathcal{H})$  denote the Banach algebra of all bounded linear operators on  $\mathcal{H}$ . Consider a subalgebra  $A \subset \mathcal{L}(\mathcal{H})$  with the following properties:

- (i)  $A$  is commutative;
- (ii)  $A$  is closed;
- (iii) If  $T \in A$  then  $T^* \in A$ ;
- (iv) The identity operator belongs to  $A$ .

Let  $\Delta$  denote the maximal ideal space of  $A$ . Since each  $T \in A$  is normal it follows that  $\|T\| = \|\hat{T}\|_\infty$  for every  $T \in A$ .

For each pair of vectors  $\xi, \eta \in \mathcal{H}$  define a mapping  $L_{\xi, \eta} : A \rightarrow \mathbb{C}$  by

$$L_{\xi, \eta}(T) = (T\xi, \eta)$$

then we have

$$|L_{\xi, \eta}(T)| \leq \|T\| \cdot \|\xi\| \|\eta\| = \|\xi\| \|\eta\| \cdot \|\hat{T}\|_\infty$$

Therefore by Theorem 1 there exists a bounded complex Radon measure  $\mu_{\xi, \eta}$  on  $\Delta$  such that  $\|\mu_{\xi, \eta}\| \leq \|\xi\| \cdot \|\eta\|$  and

$$L_{\xi, \eta}(T) = \int_\Delta \hat{T} d\mu_{\xi, \eta} \quad \text{for every } T \in A.$$

An application of the Gelfand-Neumark theorem establishes the uniqueness of the measure. The usual construction of a unique resolution of the identity on the Borel sets of  $\Delta$  can be made based on this formula. A specialization of this formula to a single normal operator leads to the classical spectral theorem. We shall not give the details here since many excellent accounts exist (c.f. Berberian [1], [2], Segal-Kunze [18]). An especially lucid presentation is given in Rudin [16].

#### BIBLIOGRAPHY

- [1] BERBERIAN, S. *Notes on spectral theory*. Van Nostrand, Princeton, N.J., 1966.
- [2] ——— *Lectures in functional analysis and operator theory*. Springer Verlag, New York-Heidelberg-Berlin, 1974.



- [3] BERG, C., J. R. C. CHRISTENSEN and P. RESSEL. Positive definite functions on abelian semigroups. *Math. Ann.* 223 (1976), pp. 253-272.
- [4] BUCY, R. S. and G. MALTESE. A representation theorem for positive functionals on involution algebras. *Math. Ann.* 162 (1966), pp. 364-367.
- [5] ——— Extreme positive definite functions and Choquet's representation theorem. *Jour. Math. Ann. Appl.* 12 (1965), pp. 371-377.
- [6] CHOQUET, G. *Lectures on analysis, vol. II*. W. A. Benjamin, Inc., New York, 1969.
- [7] ——— Deux exemples classiques de représentation intégrale. *Enseign. Math.* 15 (1969), pp. 63-75.
- [8] DIXMIER, J. *Les  $C^*$ -algèbres et leurs représentations*. Gauthier-Villars, Paris, 1964.
- [9] EBERLEIN, W. Characterizations of Fourier-Stieltjes transforms. *Duke Jour. Math.* 22 (1955), pp. 465-468.
- [10] GELFAND, I., D. RAIKOV and G. SILOV. *Commutative normed rings*. Chelsea, New York 1964.
- [11] LOOMIS, L. *Abstract harmonic analysis*. Van Nostrand, New York, 1953.
- [12] MALTESE, G. Multiplicative extensions of multiplicative functionals in Banach algebras. *Arch. Math. (Basel)* 21 (1970), pp. 502-505.
- [13] ——— Convexity methods and the Choquet boundary in Banach algebras. *Bolletino Union Math. Ital. (5)* 15-A (1978), pp. 131-136.
- [14] PHELPS, R. R. *Lectures on Choquet's theorem*. Van Nostrand, New York, 1966.
- [15] RUDIN, W. *Fourier analysis on groups*. Interscience, New York, 1962.
- [16] ——— *Functional analysis*. McGraw-Hill, New York, 1973.
- [17] SCHOENBERG, I. A remark on the preceding note by Bochner. *Bull. AMS* 40 (1934), pp. 277-278.
- [18] SEGAL, I. E. and R. KUNZE. *Integrals and operators*. McGraw-Hill, New York, 1968.

(Reçu le 15 février 1979)

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