§1. Acyclic maps and homotopy equivalences

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§ 1. ACYCLIC MAPS AND HOMOTOPY EQUIVALENCES

We will use the terminology CW-space for a space having the homotopy type of a CW-complex. The category of CW-spaces is the largest category of spaces for which the Whitehead characterization of homotopy equivalences holds.

(1.1) DEFINITION. *A space X is acyclic provided the integral reduced homology* $\tilde{H}_*(X) = 0$.

In particular, an acyclic space X is path connected, its fundamental group $\pi_1(X)$ is perfect, i.e. $\pi_1(X)$ is equal to its commutator subgroup, and for any constant coefficient module L it follows that $\tilde{H}_*(X, L) = 0$. Recall that a local coefficient system L on X is a module over $\pi_1(X)$ and that

$$H_*(X, L) = H_*(C_*(\tilde{X}) \otimes_{\mathbf{Z}\pi_1(X)} L)$$

where $C_*(\tilde{X})$ is the chain complex over $\mathbf{Z}$ viewed as a $\mathbf{Z}\pi_1(Y)$-module. In general, $\tilde{H}(X, L) \neq 0$ for an acyclic space and a local coefficient system L.

(1.2) DEFINITION/PROPOSITION. *A map $f : X \to Y$ between path connected spaces is acyclic provided any of the following equivalent conditions hold:*

(a) *The homotopy fibre F of $f : X \to Y$ is an acyclic space.*

(b) *For any local coefficient system L on Y the induced morphism*

$$f_* : H_*(X, f^* L) \to H_*(Y, L)$$

is an isomorphism where f^*L is the induced local system on X.

(c) *The induced morphism*

$$f_* : H_*(X, f^*\mathbf{Z}\pi_1(Y)) \to H_*(Y, \mathbf{Z}\pi_1(Y))$$

is an isomorphism.

(d) *For the universal covering $\tilde{Y} \to Y$ of Y the map $X \times_Y \tilde{Y} \to \tilde{Y}$ defined by f induces an isomorphism*

$$H_*(X \times_Y \tilde{Y}) \to H_*(\tilde{Y}).$$

Proof. For (a) implies (b), we use the Serre spectral sequence for the fibration $F \xrightarrow{i} X \xrightarrow{f} Y$ where

$$E^2 = H_*\left(Y, H_*\left(F, i^* f^* L\right)\right) \Rightarrow H_*\left(X, f^* L\right).$$

Since $i^* f^* L$ is trivial on F, statement (a) gives $\tilde{H}_*(F, i^* f^* L) = 0$ and the edge morphism $H_*(X, f^* L) \to H_*(Y, L) = E^2_{*,0}$, which is induced by f, is an isomorphism.

Clearly (b) implies (c), which is a special case of (b), and for (c) implies (d) we use the following morphism of fibrations

$$\begin{array}{ccc} & \pi = \pi_1(Y) & \\ \swarrow & & \searrow \\ X \times_Y \tilde{Y} & \longrightarrow & \tilde{Y} \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y. \end{array}$$

This induces a morphism of the Serre spectral sequences which on the E^2-level is the given isomorphism from (c)

$$E^2 = H_*\left(X, f^* \mathbf{Z}\pi_1(Y)\right) \to H_*\left(Y, \mathbf{Z}\pi_1(Y)\right) = E^2.$$

Hence by the spectral mapping theorem $H_*(X \times_Y \tilde{Y}) \to H_*(\tilde{Y})$ is an isomorphism.

For (d) implies (a), note that $F \to X \times_Y \tilde{Y}$ is the fibre of $X \times_Y \tilde{Y} \to \tilde{Y}$. Since $H_*(X \times_Y \tilde{Y}) \to H_*(\tilde{Y})$ is an isomorphism on the horizontal edge of the spectral sequence, we see $\tilde{H}_0(F) = 0$. Moreover, assuming inductively that $\tilde{H}_j(F) = 0$ for $i < n$, we deduce that $\tilde{H}_n(F) \doteq 0$ by looking at the spectral sequence terms $E^r_{0,n}$ which is $H_n(F)$ for $r = 2$ and zero for $r > n + 1$. This completes the proof the equivalence of (a), (b), (c), and (d).

(1.3) Proposition. *If $f : X \to Y$ is an acyclic map, then $f_* : \pi_1(X) \to \pi_1(Y)$ is an epimorphism with kernel a perfect normal subgroup.*

Proof. Since the fibre F of f is connected, the induced homomorphism f_* is an epimorphism, and since $\pi_1(F)$ is perfect, $\ker(f_*) = \operatorname{im}\left(\pi_1(F) \to \pi_1(X)\right)$ is perfect.

(1.4) PROPOSITION. *Let* $f: X \to Y$ *be a map between path connected spaces. Then* $\pi_i(f): \pi_i(X) \to \pi_i(Y)$ *is an isomorphism for all* $i \geqq 0$ *if and only if* f *is acyclic and* $\pi_1(f)$ *is an isomorphism.*

Proof. Let $F \to X$ be the homotopy fibre of f. The second conditions say that $\pi_1(F)$ is perfect and abelian respectively. Thus $\pi_1(F) = 0$ and on simply connected spaces F the homotopy $\pi_i(F) = 0$ if and only if the homology $\tilde{H}_i(F) = 0$. The proposition follows now from an application of the homotopy exact sequence.

(1.5) COROLLARY. *A map* $f: X \to Y$ *between path connected CW-spaces is a homotopy equivalence if and only if* f *is acyclic and* $\pi_1(f)$ *is an isomorphism.*

This is an immediate application of the Whitehead criterion for homotopy equivalence applied to (1.4).

In section 3 we will see that the subgroups $\ker(\pi_1(f))$ classify acyclic maps $f: X \to Y$ from X.

(1.6) *Remark.* Cohomology with local coefficients can be used to characterize acyclic maps. As with homology

$$H^*(X, L) = H^*(\mathrm{Hom}_{Z\pi_1(X)}(C^*(\tilde{X}), L))$$

defines cohomology with local coefficients. Then a map $f: X \to Y$ between path connected spaces is acyclic if and only if $f^*: H^*(Y, L) \to H^*(X, f^*L)$ is an isomorphism for each local coefficient system L on Y. The direct implication is checked exactly as (a) implies (b) using cohomology in (1.2). Conversely we show that $X \times_Y \tilde{Y} \to \tilde{Y}$ defined by f induces an isomorphism $H^*(\tilde{Y}) \to H^*(X \times_Y \tilde{Y})$. This is done as (c) implies (d) in (1.2) and as in (d) implies (a) in (1.2) we have $\tilde{H}^*(F) = 0$. Using the universal coefficient theorem, we deduce that $\tilde{H}_*(F) = 0$ and F is acyclic.

The cohomology characterization of acyclic maps is useful in obstruction theory.

(1.7) *Remark.* Let $f: X \to Y$ be an acyclic map and $\bar{Y}$ a connected covering of Y. Then the induced map $\bar{f}: X \times_Y \bar{Y} \to \bar{Y}$ is also acyclic. This follows directly from (1.2, (d)) or from the fact that f and $\bar{f}$ have the same fibre. When $\bar{Y}$ is the universal covering of Y, the space $X \times_Y \bar{Y} = \tilde{X}_N$ is the covering of X with fundamental group $N = \ker(\pi_1(f))$.