

§2. Induced and coinduced acyclic maps

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **25 (1979)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **26.05.2024**

Nutzungsbedingungen

Die ETH-Bibliothek ist Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Inhalten der Zeitschriften. Die Rechte liegen in der Regel bei den Herausgebern.

Die auf der Plattform e-periodica veröffentlichten Dokumente stehen für nicht-kommerzielle Zwecke in Lehre und Forschung sowie für die private Nutzung frei zur Verfügung. Einzelne Dateien oder Ausdrucke aus diesem Angebot können zusammen mit diesen Nutzungsbedingungen und den korrekten Herkunftsbezeichnungen weitergegeben werden.

Das Veröffentlichen von Bildern in Print- und Online-Publikationen ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Die systematische Speicherung von Teilen des elektronischen Angebots auf anderen Servern bedarf ebenfalls des schriftlichen Einverständnisses der Rechteinhaber.

Haftungsausschluss

Alle Angaben erfolgen ohne Gewähr für Vollständigkeit oder Richtigkeit. Es wird keine Haftung übernommen für Schäden durch die Verwendung von Informationen aus diesem Online-Angebot oder durch das Fehlen von Informationen. Dies gilt auch für Inhalte Dritter, die über dieses Angebot zugänglich sind.

Ein Dienst der *ETH-Bibliothek*

ETH Zürich, Rämistrasse 101, 8092 Zürich, Schweiz, www.library.ethz.ch

<http://www.e-periodica.ch>

§ 2. INDUCED AND COINDUCED ACYCLIC MAPS

(2.1) PROPOSITION. Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be two maps. If f and g are acyclic, then gf is acyclic. If f and gf are acyclic, then g is acyclic.

Proof. Consider a local system L on Z , and using g^*L on Y , $f^*g^*L = (gf)^*L$ on X , we apply (1.2) (b) to obtain the proposition.

(2.2) PROPOSITION. Consider the following cartesian square where either f or g is a fibration.

$$\begin{array}{ccc} Y' \times_Y Y & \xrightarrow{g'} & X \\ f' \downarrow & & \downarrow f \\ Y' & \xrightarrow{g} & Y \end{array}$$

If f is acyclic, then f' is acyclic.

Proof. Since either f or g is a fibration, we can change the other to be a fibration, if necessary, without changing the homotopy type of any of the four spaces. Now the homotopy fibre F of f is the actual fiber and F is also the homotopy fibre of f' . Now apply (1.2) (a).

(2.3) PROPOSITION. Consider the following cocartesian square where either f or g is a cofibration.

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ g \downarrow & & \downarrow g' \\ X' & \xrightarrow{f'} & X' \cup Y = Y' \end{array}$$

If f is acyclic, then f' is acyclic.

Proof. Since either f or g is a cofibration, we can change the other to be a cofibration, if necessary, without changing the homotopy type of any of the four spaces. Hence each map is an injection, and for a local coefficient system L on Y' , we have two long exact sequences in homology

$$\begin{array}{ccccccc} \longrightarrow H_q(X, f^*g'^*L) & \xrightarrow{f_*} & H_q(Y, f'^*L) & \longrightarrow & H_q(Y, X; f'^*L) & \longrightarrow & \dots \\ & \downarrow g_* & & & \downarrow g'^* & & \downarrow (g, g')_* \\ \longrightarrow H_q(X', g'^*L) & \xrightarrow{f'^*} & H_q(Y', L) & \longrightarrow & H_q(Y', X'; L) & \longrightarrow & \dots \end{array}$$

By hypothesis (1.2) (b) the morphism f_* is an isomorphism and thus $H_*(Y, X; f'^*L) = 0$. By excision $(g, g')_*$ is an isomorphism and thus $H_*(Y', X'; L) = 0$. Hence f'_* is an isomorphism and criterion (1.2) (b) is satisfied for f' to be an acyclic map which proves the proposition.

The previous proposition concerning acyclic maps in a cofibration will be the basic tool for most of the results which follow in sections 2 and 3. It was pointed out to us by Quillen.

(2.4) PROPOSITION. Consider the following diagram of CW-spaces.

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ g \downarrow & & \downarrow g' \\ X' & \xrightarrow{f'} & Y' \end{array}$$

If g and g' are acyclic, and if $\pi_1(f)$ and $\pi_1(f')$ are isomorphisms then the diagram is cocartesian up to homotopy equivalence.

Proof. First replace f and g by equivalent cofibrations and form $h : X' \cup_X Y \rightarrow Y'$. The map $g'' : Y \rightarrow X' \cup_X Y$ is an acyclic map by (2.3) and $g' = hg''$. Thus h is acyclic by (2.1).

Since $\pi_1(f)$ is an isomorphism, it follows that $f'' : X' \rightarrow X' \cup_X Y$ has the property that $\pi_1(f'')$ is an isomorphism by the van Kampen theorem and $f' = hf''$. Thus $\pi_1(h)$ is an isomorphism. Now apply (1.5) to see that h is a homotopy equivalence. This proves the proposition.

(2.5) THEOREM. Let $f : X \rightarrow Y$ be an acyclic map between CW-spaces with homotopy fibre $g : F \rightarrow X$. Then f is the homotopy cofibre of g .

Proof. Let CF be the cone over F . The homotopy cofibre C of $g : F \rightarrow X$ is homotopy equivalent to $CF \cup_F X$ and we have the cocartesian square

$$\begin{array}{ccccc} F & \xrightarrow{g} & X & \xrightarrow{f} & Y \\ \downarrow & & v \downarrow & & \nearrow h \\ CF & \longrightarrow & C & & \end{array}$$

Since $fg \simeq *$, it follows that we have a map $h : C \rightarrow Y$ such that $f \simeq hv$. Since f is acyclic, the map $F \rightarrow CF$ is acyclic and, by (2.3) v is acyclic. One deduces then, by (2.1) that h is acyclic. As $\pi_1(h)$ is onto (1.3), one has:

$$\ker(\pi_1(h)) = v(\ker \pi_1(f)) = v(\operatorname{Im} \pi_1(g)) = 1$$

So $\pi_1(h)$ is injective and, by (1.3) and (1.5), h is a homotopy equivalence.

(2.6) THEOREM. *Let $f: X \rightarrow Y$ be an acyclic map between CW-spaces and let $h_1, h_2: Y \rightarrow Z$ be two maps. If $h_1 f \simeq h_2 f$, then it follows that $h_1 \simeq h_2$.*

Proof. By (2.5) we have cofibre sequence

$$F \xrightarrow{g} X \xrightarrow{f} Y \longrightarrow \Delta F$$

where ΔF is the reduced suspension of the acyclic space F . Since ΔF is simply connected and $\tilde{H}_*(\Delta F) = 0$, it is contractible, and the group $[\Delta F, Z]$ in the Puppe sequence is zero.

In general, the group $[\Delta F, Z]$ acts transitively on the fibres of the function $[Y, Z] \rightarrow [X, Z]$, so that in this case, $[Y, Z] \rightarrow [X, Z]$ is injective. This proves the theorem.

§ 3. CLASSIFICATION OF ACYCLIC MAP FROM A GIVEN SPACE

Let X be a path connected space. To each acyclic map $f: X \rightarrow Y$, we assign the kernel of $\pi_1(f): \pi_1(X) \rightarrow \pi_1(Y)$ which is a perfect normal subgroup of $\pi_1(X)$ by (1.3). The object of this section is to show that this map from isomorphism classes of acyclic maps defined on X to perfect normal subgroups of $\pi_1(X)$ is a bijection.

(3.1) PROPOSITION. *Let $f: X \rightarrow Y$ and $f': X \rightarrow Y'$ be two maps between CW-spaces such that f is acyclic. There exists a map $h: Y \rightarrow Y'$ with $hf \simeq f'$ if and only if $\ker \pi_1(f) \subset \ker \pi_1(f')$, and such an h is unique up to homotopy. In addition, if f' is acyclic, then h is acyclic, and h is a homotopy equivalence if and only if $\ker \pi_1(f) = \ker \pi_1(f')$.*

Proof. If h exists, then $\pi_1(f') = \pi_1(h) \circ \pi_1(f)$ and we have $\ker \pi_1(f) \subset \ker \pi_1(f')$. Conversely, we can suppose f is a cofibration and form the cocartesian diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ f' \downarrow & & \downarrow g' \\ Y' & \xrightarrow{g} & Y' \cup_X Y \end{array}$$