§2. Induced and coinduced acyclic maps

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(2.1) Proposition. *Let $f : X \to Y$ and $g : Y \to Z$ be two maps. If f and g are acyclic, then gf is acyclic. If f and gf are acyclic, then g is acyclic.*

Proof. Consider a local system L on Z, and using g^*L on Y $f^*g^*L = (gf)^*L$ on X, we apply (1.2) (b) to obtain the proposition.

(2.2) Proposition. *Consider the following cartesian square where either f or g is a fibration.*

$$\begin{tikzcd} Y' \times_Y Y \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

If f is acyclic, then f' is acyclic.

Proof. Since either f or g is a fibration, we can change the other to be a fibration, if necessary, without changing the homotopy type of any of the four spaces. Now the homotopy fibre F of f is the actual fiber and F is also the homotopy fibre of f'. Now apply (1.2) (a).

(2.3) Proposition. *Consider the following cocartesian square where either f or g is a cofibration.*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] & Y \arrow[d, "g'"] \\ X' \arrow[r, "f'"] & X' \cup_X Y = Y' \end{tikzcd}$$

If f is acyclic, then f' is acyclic.

Proof. Since either f or g is a cofibration, we can change the other to be a cofibration, if necessary, without changing the homotopy type of any of the four spaces. Hence each map is an injection, and for a local coefficient system L on Y', we have two long exact sequences in homology

$$\begin{tikzcd} \longrightarrow H_q(X, f^*g'^*L) \arrow[r, "f_*"'] \arrow[d, "g_*"] & H_q(Y, f'^*L) \arrow[r] \arrow[d, "g'_*"] & H_q(Y, X; f'^*L) \arrow[r] \arrow[d, "(g, g')_*"] & \ldots \\ \longrightarrow H_q(X', g'^*L) \arrow[r, "f'_*"] & H_q(Y', L) \arrow[r] & H_q(Y', X'; L) \arrow[r] & \ldots . \end{tikzcd}$$

By hypothesis (1.2) (b) the morphism f_* is an isomorphism and thus $H_*(Y, X; f'^*L) = 0$. By excision $(g, g')_*$ is an isomorphism and thus $H_*(Y', X'; L) = 0$. Hence f'_* is an isomorphism and criterion (1.2) (b) is satisfied for f' to be an acyclic map which proves the proposition.

The previous proposition concerning acyclic maps in a cofibration will be the basic tool for most of the results which follow in sections 2 and 3. It was pointed out to us by Quillen.

(2.4) PROPOSITION. *Consider the following diagram of CW-spaces.*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] & Y \arrow[d, "g'"] \\ X' \arrow[r, "f'"] & Y' \end{tikzcd}$$

If g and g' are acyclic, and if $\pi_1(f)$ and $\pi_1(f')$ are isomorphisms then the diagram is cocartesian up to homotopy equivalence.

Proof. First replace f and g by equivalent cofibrations and form $h : X' \cup_X Y \to Y'$. The map $g'' : Y \to X' \cup_X Y$ is an acyclic map by (2.3) and $g' = hg''$. Thus h is acyclic by (2.1).

Since $\pi_1(f)$ is an isomorphism, it follows that $f'' : X' \to X' \cup_X Y$ has the property that $\pi_1(f'')$ is an isomorphism by the van Kampen theorem and $f' = hf''$. Thus $\pi_1(h)$ is an isomorphism. Now apply (1.5) to see that h is a homotopy equivalence. This proves the proposition.

(2.5) THEOREM. *Let $f : X \to Y$ be an acyclic map between CW-spaces with homotopy fibre $g : F \to X$. Then f is the homotopy cofibre of g.*

Proof. Let CF be the cone over F. The homotopy cofibre C of $g : F \to X$ is homotopy equivalent to $CF \cup_F X$ and we have the cocartesian square

$$\begin{tikzcd} F \arrow[r, "g"] \arrow[d] & X \arrow[r, "f"] \arrow[d, "v"] & Y \\ CF \arrow[r] & C \arrow[ur, dashed, "h"] & \end{tikzcd}$$

Since $fg \simeq *$, it follows that we have a map $h : C \to Y$ such that $f \simeq hv$. Since f is acyclic, the map $F \to CF$ is acyclic and, by (2.3) v is acyclic. One deduces then, by (2.1) that h is acyclic. As $\pi_1(h)$ is onto (1.3), one has:

$$\ker(\pi_1(h)) = v(\ker \pi_1(f)) = v(\mathrm{Im}\, \pi_1(g)) = 1$$

So $\pi_1(h)$ is injective and, by (1.3) and (1.5), h is a homotopy equivalence.

(2.6) THEOREM. *Let $f: X \to Y$ be an acyclic map between CW-spaces and let $h_1, h_2 : Y \to Z$ be two maps. If $h_1 f \simeq h_2 f$, then it follows that $h_1 \simeq h_2$.*

Proof. By (2.5) we have cofibre sequence

$$F \xrightarrow{g} X \xrightarrow{f} Y \longrightarrow \Delta F$$

where ΔF is the reduced suspension of the acyclic space F. Since ΔF is simply connected and $\tilde{H}_*(\Delta F) = 0$, it is contractible, and the group $[\Delta F, Z]$ in the Puppe sequence is zero.

In general, the group $[\Delta F, Z]$ acts transitively on the fibres of the function $[Y, Z] \to [X, Z]$, so that in this case, $[Y, Z] \to [X, Z]$ is injective. This proves the theorem.

§ 3. CLASSIFICATION OF ACYCLIC MAP FROM A GIVEN SPACE

Let X be a path connected space. To each acyclic map $f: X \to Y$, we assign the kernel of $\pi_1(f) : \pi_1(X) \to \pi_1(Y)$ which is a perfect normal subgroup of $\pi_1(X)$ by (1.3). The object of this section is to show that this map from isomorphism classes of acyclic maps defined on X to perfect normal subgroups of $\pi_1(X)$ is a bijection.

(3.1) PROPOSITION. *Let $f: X \to Y$ and $f': X \to Y'$ be two maps between CW-spaces such that f is acyclic. There exists a map $h: Y \to Y'$ with $hf \simeq f'$ if and only if* $\ker \pi_1(f) \subset \ker \pi_1(f')$, *and such an h is unique up to homotopy. In addition, if f' is acyclic, then h is acyclic, and h is a homotopy equivalence if and only if* $\ker \pi_1(f) = \ker \pi_1(f')$.

Proof. If h exists, then $\pi_1(f') = \pi_1(h) \circ \pi_1(f)$ and we have $\ker \pi_1(f) \subset \ker \pi_1(f')$. Conversely, we can suppose f is a cofibration and form the cocartesian diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle g'} \\ Y' & \xrightarrow{g} & Y' \cup_X Y \end{array}$$