INVARIANT SOLUTIONS OF ANALYTIC EQUATIONS

INVARIANT SOLUTIONS OF ANALYTIC EQUATIONS

by Edward BIERSTONE and Pierre MILMAN

1. INTRODUCTION

Let $\mathbf{k}$ be a field of characteristic zero with a non-trivial valuation.

We consider a system of analytic equations

$$(*) \qquad f(x, y) = 0,$$

where

$$f(x, y) = (f_1(x, y), \ldots, f_q(x, y))$$

are convergent series in the variables

$$x = (x_1, \ldots, x_n),$$
$$y = (y_1, \ldots, y_p).$$

Suppose that

$$\bar{y}(x) = (\bar{y}_1(x), \ldots, \bar{y}_p(x)), \quad \bar{y}_j(x) \in \mathbf{k}[[x]],$$

are formal power series without constant term which solve (*); i.e. such that $f(x, \bar{y}(x)) = 0$. Let c be a non-negative integer. Artin's approximation theorem [3] asserts that there exists a convergent series solution

$$y(x) = (y_1(x), \ldots, y_p(x)), \quad y_j(x) \in \mathbf{k}\{x\},$$

of (*), such that

$$y(x) \equiv \bar{y}(x) \bmod \mathfrak{m}^c.$$

Here $\mathfrak{m}$ denotes the maximal ideal of $\mathbf{k}[[x]]$.

Artin also proved an algebraic analogue of this theorem [1]. It says that if $f(x, y) = 0$ is a system of polynomial equations with formal series solution $\bar{y}(x)$, then a series solution $y(x)$ may be found such that the $y_j(x)$ are algebraically dependent on $x_1, \ldots, x_n$ (we will say that the $y_j(x)$ are "algebraic"; cf. [2]). In this analogue $\mathbf{k}$ is an arbitrary field.

Let G be a reductive algebraic group (i.e. G is linear and every rational representation of G is completely reducible). Suppose that G acts linearly on $V = \mathbf{k}^n$ and $W = \mathbf{k}^p$. We will say that $\bar{y}(x) \in \mathbf{k}[[x]]^p$ is *equivariant* if

$$\bar{y}(gx) = g\bar{y}(x), \quad g \in G.$$

We will prove the following theorem.

THEOREM A. *Suppose* $\mathbf{k} = \mathbf{R}$ *or* $\mathbf{C}$, *and that* $\bar{y}(x) \in \mathbf{k}[[x]]^p$ *is an equivariant formal power series solution of* (*), $\bar{y}(0) = 0$. *Let* $c \in \mathbf{N}$. *Then there exists an equivariant convergent series solution* $y(x)$ *of* (*), *such that* $y(x) \equiv \bar{y}(x) \bmod \mathfrak{m}^c$.

Moreover, if $f(x, y) = 0$ *is a system of polynomial equations (where* $\mathbf{k}$ *is any field), then there exists an equivariant algebraic solution* $y(x)$, *such that* $y(x) \equiv \bar{y}(x) \bmod \mathfrak{m}^c$.

Remark 1.1. Theorem A may be regarded in the context of the question: What properties of a formal solution of (*) may be preserved in an analytic solution? Artin [2] asked whether there is a convergent solution such that some of the variables x_i are missing in some of the series $y_j(x)$, provided there is a formal solution with the same property. Gabrielov [6] answered this question negatively (see also [4]). In [12] it is shown that if a formal solution of a system of real analytic equations satisfies the Cauchy-Riemann equations, then it may be approximated by complex analytic solutions.

Remark 1.2. Suppose that $\pi(x) \in \mathbf{C}\{x\}^r$ is an analytically regular germ of an analytic mapping (terminology of Gabrielov [7]). Let $F_i(x) \in \mathbf{C}\{x\}^p$, $i = 1, \ldots, q$. We may ask whether formal relations among the F_i of the form $(h_1(\pi(x)), \ldots, h_q(\pi(x)))$; i.e. q-tuples of formal power series of this form such that

$$\sum_{i=1}^{q} h_i(\pi(x)) F_i(x) = 0,$$

are generated by analytic relations of the same form. This question generalizes Gabrielov's problem in [7]. The answer is *no* in general, but the method of our proof of Theorem A shows it is *yes* if π is a finite analytic germ. As in our proof of Theorem A, it is then easy to see that a formal solution $\bar{y}(\pi(x))$ of a system of complex analytic equations $f(x, y) = 0$ may be approximated by analytic solutions of the same form. We are grateful to Joseph Becker for pointing out the latter result to us.

Remark 1.3. Tougeron [16] has proved a generalization of Artin's theorem which asserts, in particular, that every formal solution $\bar{y}(x)$ of (*) such that $\bar{y}(0) = 0$ is the formal Taylor series at 0 of an infinitely differentiable solution. The proof of Theorem A also gives an equivariant version of Tougeron's theorem.

Theorem A is closely related to the second result of this paper.

THEOREM B. *Suppose that G acts linearly on $V = \mathbf{k}^n$, and that X is a closed algebraic subset of V which is invariant under the action of G. Then there exists a linear action of G on a finite dimensional vector space $Y = \mathbf{k}^q$, and an equivariant polynomial mapping $F: V \to Y$ such that $X = F^{-1}(0)$.*

If $\mathbf{k} = \mathbf{R}$ or $\mathbf{C}$, and X is a germ at 0 of a closed analytic subset of V which is invariant under the action of G, then there exists a vector space $Y = \mathbf{k}^q$ on which G acts linearly, and a germ F of an equivariant analytic mapping of some neighborhood if $0 \in V$ into Y, such that $X = F^{-1}(0)$.

A linear action of G on $\mathbf{k}^n$ induces an action on $\mathbf{k}[[x]] = \mathbf{k}[[x_1, ..., x_n]]$ (respectively $\mathbf{k}\{x\}$, $\mathbf{k}[x]$) such that

$$(g \cdot f)(x) = f(g^{-1}x)$$

for all $g \in G$ and $f(x) \in \mathbf{k}[[x]]$ (respectively $\mathbf{k}\{x\}$, $\mathbf{k}[x]$). Let $\mathbf{k}[[x]]^G$ (respectively $\mathbf{k}\{x\}^G$, $\mathbf{k}[x]^G$) be the subset of elements fixed by G (the *invariant* elements).

Remark 1.4. It is well-known that if $\mathbf{k} = \mathbf{R}$ and G is compact, then the conclusion of Theorem B holds with $F \in (\mathbf{R}[x]^G)^q$ (or $F \in (\mathbf{R}\{x\}^G)^q$ in the analytic case). In general, invariants separate only disjoint Zariski closed invariant subsets of $\mathbf{k}^n$, so that invariant closed algebraic or analytic subsets needn't be defined by invariant equations.

We will prove Theorem B in the following section, considering separately the complex analytic, real analytic, and algebraic cases. These results may also be obtained in a unified way, at least in characteristic zero, from an explicit projection formula related to the Fourier transform (cf. [15], [10, 12.2]). This formula may be of independent interest, and we have included it in section 4. In section 3 we will deduce Theorem A from Theorem B.

The authors enjoyed several conversations with Joseph Becker on the results in this paper.

2. Proof of Theorem B

2.1. The complex analytic case. Let G be a reductive complex algebraic group. Then G is the universal complexification of a compact real Lie group $G^{\mathbf{R}}$ [9], [8, XVII.5].

Suppose that G acts linearly on $V = \mathbf{C}^n$, and that X is a germ at 0 of an invariant closed analytic subset of V. Let I be the ideal in $\mathbf{C}\{x\} = \mathbf{C}\{x_1, ..., x_n\}$ of germs of holomorphic functions which vanish on X. Suppose that I is generated by $f_1, ..., f_k$.

For any irreducible complex representation $T: G \to GL(W)$ of G, we consider the action of G on the space $\operatorname{End}_{\mathbf{C}}(W, W)$ of complex linear endomorphisms defined by

$$(g \cdot \lambda)(w) = T(g)\,\lambda(w),$$

where $g \in G$, $w \in W$ and $\lambda \in \operatorname{End}_{\mathbf{C}}(W, W)$. For each $i = 1, ..., k$, we consider the mapping

$$f_i^T : V \to \operatorname{End}_{\mathbf{C}}(W, W)$$

$$f_i^T(x) = \int_{G^{\mathbf{R}}} f_i(g^{-1}x)\, T(g)\, dg\,,$$

defined in an open neighborhood of 0 where f_i converges. Then f_i^T is equivariant with respect to the actions of $G^{\mathbf{R}}$ on V and $\operatorname{End}_{\mathbf{C}}(W, W)$, and hence with respect to the actions of G (the "unitarian trick"). Furthermore $f_i(gx) = 0$ for all $g \in G$ if and only if $f_i^T(x) = 0$ for all irreducible complex representations T of $G^{\mathbf{R}}$ (cf. [10, 12.2]; this is essentially the Peter-Weyl theorem).

Hence X is defined by the equations

$$f_i^T(x) = 0\,,$$

where $1 \leqslant i \leqslant k$ and T runs over all irreducible complex representations of $G^{\mathbf{R}}$. It follows that X is defined by a finite subset of these equivariant equations.

2.2. The real analytic case. Let G be a reductive real algebraic group. Then the universal complexification $G^{\mathbf{C}}$ of G is a reductive complex algebraic group [8, XVIII.4].

Suppose that G acts linearly on $V = \mathbf{R}^n$, and that X is a germ at 0 of an invariant real analytic subset of V. The complexification $X^{\mathbf{C}}$ of X is a germ at 0 of a complex analytic subset of $V^{\mathbf{C}} = \mathbf{C}^n$. The complexification $X^{\mathbf{C}}$ is invariant under the induced action of $G^{\mathbf{C}}$ on $V^{\mathbf{C}}$.

By the complex analytic case 2.1, there is a linear action of $G^{\mathbf{C}}$ on a finite dimensional complex vector space W, and a germ H at 0 of a $G^{\mathbf{C}}$-equivariant holomorphic mapping of some neighborhood of $0 \in V^{\mathbf{C}}$ into W, such that $X^{\mathbf{C}} = H^{-1}(0)$.

Let Y be W with its underlying real structure. Then $F = H \mid V : V \to Y$ is G-equivariant, and $X = F^{-1}(0)$.

2.3. The algebraic case. Our ground field $\mathbf{k}$ is now arbitrary. Let G be a reductive algebraic group acting linearly on $V = \mathbf{k}^n$, and let X be an invariant algebraic subset of V. Let I be the ideal in $\mathbf{k}[x]$ of polynomials which vanish on X, and $\mathbf{k}[x]_c$ be the subspace of $\mathbf{k}[x]$ of polynomials of degree at most c. Then I and $\mathbf{k}[x]_c$ are invariant subsets of $\mathbf{k}[x]$.

For each $c \in \mathbf{N}$, we define a polynomial mapping

$$F_c : V \to \operatorname{End}_{\mathbf{k}}\left(I \cap \mathbf{k}[x]_c, \mathbf{k}\right)$$

by the formula $F_c(x)(h) = h(x)$, where $x \in V$ and $h \in I \cap \mathbf{k}[x]_c$. Then F_c is equivariant and $X \subset F_c^{-1}(0)$ for all $c \in \mathbf{N}$.

We consider the ideal J in $\mathbf{k}[x]$ generated by the coordinate functions of all equivariant polynomial mappings defined on V, which vanish on X. Since J is finitely generated, it suffices to show that $J = I$. Clearly $J \subset I$. On the other hand, suppose $h \in I \cap \mathbf{k}[x]_c$, $h \neq 0$. Let $\{e_j\}_{1 \leqq j \leqq q}$ be a basis of the vector space $I \cap \mathbf{k}[x]_c$, such that $e_1 = h$. Then h is the first coordinate function of the equivariant mapping F_c, with respect to the dual basis $\{e_j^*\}_{1 \leqq j \leqq q}$ in $\operatorname{End}_{\mathbf{k}}(I \cap \mathbf{k}[x]_c, \mathbf{k})$. Since $X \subset F_c^{-1}(0)$, then $h \in J$. Hence $J = I$ as required.

This case of Theorem B may also be obtained from a lemma of Cartier [13, p. 25].

3. Proof of Theorem A

The formal power series $\bar{y}(x) \in \mathbf{k}[[x]]^p$ define a local $\mathbf{k}$-homomorphism $\phi : \mathbf{k}\{x, y\} \to \mathbf{k}[[x]]$ (or a $\mathbf{k}$-homomorphism $\phi : \mathbf{k}[x, y] \to \mathbf{k}[[x]]$ in the algebraic case) by substitution: $h(x, y) \to h\left(x, \bar{y}(x)\right)$.

Let X be the germ at 0 in $V \times W = \mathbf{k}^{n+p}$ of a closed analytic subset (or the closed algebraic subset of $V \times W$ in the algebraic case) defined by the prime ideal $\ker \phi$. It follows from Artin's approximation theorem that $\ker \phi$ satisfies the nullstellensatz (whether or not $\mathbf{k}$ is algebraically closed). In other words, if $h(x, y)$ vanishes on X, then $h(x, y) \in \ker \phi$. In fact if h vanishes on X, then for any $c \in \mathbf{N}$ we can find a convergent series solution $y(x)$ of the system of equations determined by the ideal $\ker \phi$, such that $y(x) \equiv \bar{y}(x) \bmod \mathfrak{m}^c$. Then $h\,(x, y(x)) = 0$ and $h(x, y(x)) \equiv h(x, \bar{y}(x)) \bmod \mathfrak{m}^c$. Hence $h \in \ker \phi$.

It follows that Theorem B reduces Theorem A to the case of an equivariant equation. We may assume that $f(x, y) \in \mathbf{k}\{x, y\}^q$ (respectively $f(x, y) \in \mathbf{k}[x, y]^q$) is the germ of an equivariant analytic mapping (respectively the equivariant polynomial mapping) given by Theorem B for the invariant analytic set germ (respectively algebraic set) X.

From now on, then, we assume that G acts linearly on $V = \mathbf{k}^n$, $W = \mathbf{k}^p$ and $Y = \mathbf{k}^q$, and that $f(x, y)$ is a germ of an equivariant analytic mapping (or an equivariant polynomial mapping in the algebraic case).

Since G is reductive, then $\mathbf{k}[x]^G$ is finitely generated (as a $\mathbf{k}$-algebra) by homogeneous polynomials $u_1(x), \ldots, u_r(x) \in \mathbf{k}[x]^G$ [13, Theorem 1.1]. Hence the homomorphisms

$$u^*: \mathbf{k}[u] \to \mathbf{k}[x]^G,$$
$$\hat{u}^*: \mathbf{k}[[u]] \to \mathbf{k}[[x]]^G$$

defined by substitution $h(u_1, \ldots, u_r) \to h(u_1(x), \ldots, u_r(x))$ are surjective. If $\mathbf{k} = \mathbf{R}$ or $\mathbf{C}$, then the induced homomorphism

$$\tilde{u}^*: \mathbf{k}\{u\} \to \mathbf{k}\{x\}^G$$

is surjective by a result of Luna [11].

In the remainder of the proof we consider only the analytic case. The proof of the algebraic case is identical, if we replace the analytic version of Artin's approximation theorem by the algebraic version.

Remark 3.1. If G acts trivially on Y, i.e. $f_j(x, y) \in \mathbf{k}\{x, y\}^G$, $j = 1, \ldots, q$, then our theorem follows immediately. In fact let $I = \ker u^*$. Then $\ker \hat{u}^* = I \cdot \mathbf{k}[[u]]$ and $\ker \tilde{u}^* = I \cdot \mathbf{k}\{u\}$ (the former equality follows by expressing a power series in $\mathbf{k}[[u]]$ as a sum of weighted homogeneous polynomials, weighted by the degrees of the u_i, and the latter then by Artin's theorem). Suppose that $F_1(x), \ldots, F_s(x)$ generate the module of equivariant polynomial mappings of V into W over the ring $\mathbf{k}[x]^G$ of invariant polynomials on V. Since $f(x, \Sigma_{i=1}^{s} \eta_i F_i(x))$ is invariant in (x, η), where G acts trivially on the variables $\eta = (\eta_1, \ldots, \eta_s)$, then there exists $h \in \mathbf{k}[u, \eta]^q$, such that

$$f\left(x, \sum_{i=1}^{s} \eta_i F_i(x)\right) = h(u(x), \eta).$$

If $\bar{y}(x) = \sum_{i=1}^{s} \bar{\eta}_i(u(x)) F_i(x)$ is a formal solution of $f(x, y) = 0$, then

$$h(u, \bar{\eta}(u)) \in I \cdot \mathbf{k}[[u]]^q.$$

By Artin's theorem we may approximate $\bar{\eta}(u)$ by a convergent $\eta(u)$ such that

$$h(u, \eta(u)) \in I \cdot \mathbf{k}\{u\}^q.$$

Then

$$y(x) = \sum_{i=1}^{s} \eta_i(u(x)) F_i(x)$$

is an analytic solution of $f(x, y) = 0$, approximating $\bar{y}(x)$.

In general, suppose that $F_1(x), \ldots, F_s(x)$ (respectively $G_1(x), \ldots, G_t(x)$) generate the module of equivariant polynomial mappings of V into W (respectively of V into Y), over the ring $\mathbf{k}[x]^G$. Then we may write

$$f\left(x, \sum_{i=1}^{s} \eta_i F_i(x)\right) = \sum_{j=1}^{t} h_j(u(x), \eta) G_j(x),$$

where $h_j(u, \eta) \in \mathbf{k}\{u, \eta\}$, $j = 1, \ldots, t$. (This may be proved, for example, in the same way as Proposition 3.2 of [5]).

Let M (respectively $\hat{M}$) be the $\mathbf{k}[u]$ − (respectively $\mathbf{k}[[u]]$ −) sub-module of $\mathbf{k}[u]^t$ (respectively $\mathbf{k}[[u]]^t$) of t-tuples $(h_1(u), \ldots, h_t(u))$ such that

$$\sum_{j=1}^{t} h_j(u(x)) G_j(x) = 0.$$

Suppose that M is generated by $h^k(u) = (h_1^k(u), \ldots, h_t^k(u))$, $k = 1, \ldots, m$. Then $\hat{M} = \mathbf{k}[[u]] \cdot M$. To see this, we may assume that $G_j(x)$ is homogeneous, of degree d_j say. Let $h(u) = (h_1(u), \ldots, h_t(u)) \in \hat{M}$. We write

$$h_j(u) = \sum_{l} h_{jl}(u),$$

where h_{jl} is weighted homogeneous (weighted by the degrees of the polynomials $u_i(x)$) of degree $l - d_j$. Then

$$\sum_{j=1}^{t} h_{jl}\big(u(x)\big)\, G_j(x) = 0$$

for each l; i.e. $\big(h_{1l}(u), \ldots, h_{tl}(u)\big) \in M$. Hence we may write

$$\big(h_{1l}(u), \ldots, h_{tl}(u)\big) = \sum_{k=1}^{m} \phi_l^k(u)\, h^k(u),$$

where $\phi_l^k(u) \in \mathbf{k}[u]$, so that

$$h(u) = \sum_{k=1}^{m} \Big(\sum_l \phi_l^k(u)\Big) h^k(u)$$

as required.

Now suppose that $\bar{y}(x) = \Sigma_{i=1}^{s} \bar{\eta}_i\big(u(x)\big) F_i(x)$ is a formal solution of $f(x, y) = 0$; i.e.

$$\Big(h_1\big(u, \bar{\eta}(u)\big), \ldots, h_t\big(u, \bar{\eta}(u)\big)\Big) \in \hat{M},$$

or

$$h_j\big(u, \bar{\eta}(u)\big) = \sum_{k=1}^{m} \bar{\phi}^k(u)\, h_j^k(u), \quad 1 \leqslant j \leqslant t,$$

where $\bar{\phi}^k(u) \in \mathbf{k}[[u]]$, $1 \leqslant k \leqslant m$. Then by Artin's theorem there are convergent power series $\eta(u)$, $\phi^k(u)$, such that

$$h_j\big(u, \eta(u)\big) = \sum_{k=1}^{m} \phi^k(u)\, h_j^k(u), \quad 1 \leqslant j \leqslant t,$$

and $\eta(u) \equiv \bar{\eta}(u)$, $\phi^k(u) \equiv \bar{\phi}^k(u) \bmod m^c$. Let

$$y(x) = \sum_{i=1}^{s} \eta_i\big(u(x)\big) F_i(x).$$

Then $y(x)$ is equivariant, $y(x) \equiv \bar{y}(x) \bmod m^c$, and

$$f\Big(x, y(x)\Big) = f\Big(x, \sum_{i=1}^{s} \eta_i\big(u(x)\big) F_i(x)\Big)$$

$$= \sum_{j=1}^{t} h_j\Big(u(x), \eta\big(u(x)\big)\Big) G_j(x) = 0.$$

Remark 3.2. There are more precise formulations of Artin's approximation theorem (due to Artin [1] in the algebraic case, and John Wavrik [17] in the analytic case) which assert that for every positive integer α there is a positive integer $\beta(\alpha)$ such that for each $\beta \geqslant \beta(\alpha)$, every β-order formal solution $\bar{y}(x)$ of $f(x, y) = 0$ (i.e. $\bar{y}(x)$ such that $f(x, \bar{y}(x)) \equiv 0 \bmod \mathfrak{m}^{\beta+1}$) may be approximated to order α by an algebraic or convergent solution. The method of our proof of Theorem B also provides invariant versions of these results. The one point worth noting is that for every positive integer γ, there exists a positive integer $\beta(\gamma)$ such that if $\bar{\eta}(u(x))$ is a $\beta(\gamma)$-order solution of

$$\sum_{j=1}^{t} h_j(u(x), \eta)\, G_j(x) = 0$$

(we are using the above notation), then there exist $\bar{\phi}^k(u)$, $k = 1, \ldots, m$, such that $(\bar{\eta}(u), \bar{\phi}(u))$ is a γ-order solution of

$$h_j(u, \eta) = \sum_{k=1}^{m} \phi^k h_j^k(u), \quad 1 \leqslant j \leqslant t.$$

This statement follows from a simple extension of a theorem of Chevalley [14, 30.1].

4. A projection formula

Let G be a compact Lie group and $M = L^2(G, dg)$ the space of complex-valued functions on G which are square integrable with respect to the normalized Haar measure dg. The mapping $f \to f^T$ from M into a space of continuous matrix-valued functions on G, defined for each irreducible complex representation T of G by the formula

$$\begin{aligned} f^T(h) &= \int_G f(g^{-1}h)\, T(g)\, dg \\ &= T(h) \cdot \int_G f(g^{-1})\, T(g)\, dg, \end{aligned}$$

where $h \in G$, is a generalized Fourier transform [10, Section 12] (cf. our proof of Theorem B in the complex analytic case). The Peter-Weyl theorem gives

$$f(h) = \sum_T \dim T \cdot \operatorname{tr} f^T(h),$$

where the sum is taken over all finite dimensional inequivalent irreducible complex representations T. Moreover, the mapping $\pi_T : M \to M$ defined by

$$(\pi_T f)(h) = \dim T \cdot \operatorname{tr} f^T(h),$$

where $h \in G$, is the projection onto the largest invariant subspace of M whose irreducible invariant subspaces are all equivalent to the representation space of T.

Now let G be a reductive algebraic group defined over a field $\mathbf{k}$ of characteristic zero. A vector space M on which G acts linearly will be called a *G-module*. We will obtain projection formulas similar to the above in the following cases:

(a) M is a finite dimensional G-module;

(b) $M = \mathbf{k}[x]$ or $\mathbf{k}[[x]]$;

(c) $M = \mathbf{k}\{x\}$, with $\mathbf{k} = \mathbf{R}$ or $\mathbf{C}$;

where, in cases (b) and (c), $x = (x_1, ..., x_n)$ denotes a coordinate system in a finite dimensional G-module V, and M has the induced action of G.

If L, M are G-modules, then the space $M^L = \operatorname{End}_{\mathbf{k}}(L, M)$ of $\mathbf{k}$-linear mappings $A : L \to M$ is a G-module, with the action of G defined by $g \cdot A = gAg^{-1}$. If L is an irreducible G-module, then $\mathbf{F}^L = \operatorname{End}_{\mathbf{k}}(L, L)^G$ is a field (in general not commutative). It is clear that $\mathbf{k}$ is a subfield of $\mathbf{F}^L$, and that the action of G on L commutes with the multiplication of elements of L by elements of $\mathbf{F}^L$.

We define a $\mathbf{k}$-homomorphism

$$J : \mathbf{F}^L \to \operatorname{End}_{\mathbf{k}}(\mathbf{F}^L, \mathbf{F}^L)$$

by

$$J(\lambda)(\mu) = \lambda \cdot \mu,$$

where λ, $\mu \in \mathbf{F}^L$, and let

$$\operatorname{tr}_L : \operatorname{End}_{\mathbf{k}}(L, L) \to \mathbf{k},$$

$$\operatorname{tr}_{\mathbf{F}^L} : \operatorname{End}_{\mathbf{k}}(\mathbf{F}^L, \mathbf{F}^L) \to \mathbf{k}$$

be the trace homomorphisms. It is not difficult to check that

$$\operatorname{tr}_L(\lambda) = m_L \operatorname{tr}_{\mathbf{F}^L}(J(\lambda))$$

for all $\lambda \in \mathbf{F}^L$, where m_L is the dimension of L over $\mathbf{F}^L$.

For each $v^* \in \mathrm{End}_{\mathbf{k}}(L, \mathbf{k})$ and $f \in M$, we denote by $v^* \otimes f \in \mathrm{End}_{\mathbf{k}}(L, M)$ the mapping $(v^* \otimes f)(w) = v^*(w) \cdot f$, $w \in L$. We also define a generalized trace homomorphism

$$\mathrm{Tr} : \mathrm{End}_{\mathbf{F}^L}(L, \mathbf{F}^L) \to \mathrm{End}_{\mathbf{k}}(L, \mathbf{k})$$

by the formula

$$(\mathrm{Tr}\, v^{\#})(w) = \mathrm{tr}_{\mathbf{F}^L} J(v^{\#}(w)),$$

where $v^{\#} \in \mathrm{End}_{\mathbf{F}^L}(L, \mathbf{F}^L)$ and $w \in L$.

In the following, E_M will denote a Reynolds operator for a G-module M; i.e. E_M is an invariant projection operator from M onto M^G [13, Definition 1.5].

Proposition 4.1. *Suppose L is a finite dimensional irreducible G-module. Let $\{v_{j,L}\}_{1 \leq j \leq m_L}$ be a basis for L over $\mathbf{F}^L$, and $\{v^{\#}_{j,L}\}_{1 \leq j \leq m_L}$ be its dual basis. We consider one of the following G-modules M:*

(a) *M is a finite dimensional G-module;*

(b) $M = \mathbf{k}[x]$ or $\mathbf{k}[[x]]$;

(c) $M = \mathbf{k}\{x\}$, $\mathbf{k} = \mathbf{R}$ or $\mathbf{C}$.

(In the latter two cases, the action of G is induced by a linear action on the space of coordinates $x = (x_1, \ldots, x)$.) We define $\pi_L \in \mathrm{End}_{\mathbf{k}}(M, M)$ by

$$\pi_L(f) = m_L \sum_{j=1}^{m_L} E_{M^L}(\mathrm{Tr}\, v^{\#}_{j,L} \otimes f)(v_{j,L}),$$

where $f \in M$. Then

(1) *π_L is a projection from M onto an invariant subspace whose irreducible invariant subspaces are all equivalent to L;*

(2) *for each $f \in M$.*

$$f = \sum_L \pi_L(f),$$

where the sum is taken over all finite dimensional inequivalent irreducible G-modules L (in cases (b) and (c) the sum converges in the Krull topology);

(3) *if I is an invariant ideal in M, in cases (b) and (c), then $\pi_L(f) \in I$ and*

$$E_{M^L}(\operatorname{Tr} v^{\#}_{j,L} \otimes f) \in \operatorname{End}_{\mathbf{k}}(L, I)$$

for all $f \in I$.

Remark 4.2. For each of the G-modules M of Proposition 4.1, there is a unique Reynolds operator E_{M^L}, and the mapping $M \to E_{M^L}$ is functorial. If M is finite dimensional, then this follows from the definition of "reductive". If $M = \mathbf{k}[x]$ or $\mathbf{k}[[x]]$ it follows from Cartier's lemma [13, p. 25]. If $M = \mathbf{C}\{x\}$ we define

$$E_{M^L}(f) = \int_H h \cdot f \, dh,$$

where $f \in M^L$ and H is a maximal compact subgroup of G. Finally if $M = \mathbf{R}\{x\}$, we put $E_{M^L}(f) = \operatorname{Re} E(f)$ for $f \in M^L$, where E is the Reynolds operator for the action of the complexification $G^{\mathbf{C}}$ of G on $\mathbf{C} \otimes_{\mathbf{R}} M^L$, and $\operatorname{Re} : \mathbf{C} \otimes_{\mathbf{R}} M^L \to M^L$ is the mapping $\operatorname{Re}(f) = \frac{1}{2}(f + \bar{f})$.

Remark 4.3. Proposition 4.1 provides an alternative proof of Theorem B when char $\mathbf{k} = 0$. Let I be the ideal in $\mathbf{k}[x]$ of an invariant algebraic subset of $\mathbf{k}^n$ (respectively the ideal in $\mathbf{k}\{x\}$ of a germ at 0 of an invariant analytic subset of $\mathbf{k}^n$, $\mathbf{k} = \mathbf{R}$ or $\mathbf{C}$). Then for each $f \in I$ and $v^{\#} \in \operatorname{End}_{\mathbf{F}^L}(L, \mathbf{F}^L)$, we define a polynomial mapping (respectively a germ at 0 of an analytic mapping)

$$F_{f, v^{\#}} : \mathbf{k}^n \to \operatorname{End}_{\mathbf{k}}(L, \mathbf{k})$$

by the formula

$$F_{f, v^{\#}}(x)(w) = (E_{M^L}(\operatorname{Tr} v^{\#} \otimes f)(w))(x),$$

where $x \in \mathbf{k}^n$ and $w \in L$. Then $F_{f, v^{\#}}$ is equivariant and $X \subset F^{-1}_{f, v^{\#}}(0)$. We may now argue as in our proof of the algebraic case 2.3 of Theorem B. We use the facts that $(E_{M^L}(\operatorname{Tr} v^{\#}_{j,L} \otimes f)(v_{j,L}))(x)$ is a coordinate function of $F_{f, v^{\#}_{j,L}}$ and that $\Sigma_L \pi_L(f)$ converges to f in the Krull topology, to show that the ideal I coincides with the ideal in $\mathbf{k}[x]$ (respectively $\mathbf{k}\{x\}$) generated by the coordinate functions of all equivariant polynomial mappings (respectively germs at 0 of equivariant analytic mappings) F such that $X \subset F^{-1}(0)$.

Proof of Proposition 4.1. We first consider the case (a) that M is a finite dimensional G-module. We write M as a direct sum $M = \oplus_L M_L$ of G-submodules M_L, where the sum is taken over inequivalent irreducible G-submodules L, in such a way that each nonzero irreducible G-submodule of M_L is equivalent to L. Let $f = \Sigma_L f_L$, where $f_L \in M_L$. It is enough to prove that $\pi_L f = f_L$; in other words that $\pi_L f_{L'} = 0$ if $L \neq L'$, and $\pi_L f_L = f_L$.

The first condition follows from the fact that $\mathrm{End}_{\mathbf{k}}(L, L')^G = 0$. Using the functorial property of the Reynolds operators, we reduce the second to the case $M = L$; i.e. we must prove $\pi_L f = f$ for all $f \in L$. Since

$$f = \sum_{j=1}^{m_L} v^{\#}_{j,L}(f) \cdot v_{j,L},$$

it is enough to show that

$$m_L \cdot E_{LL}(\mathrm{Tr}\, v^{\#} \otimes f) = v^{\#}(f)$$

for all $f \in L$ and $v^{\#} \in \mathrm{End}_{\mathbf{F}^L}(L, \mathbf{F}^L)$.

For each $\beta \in \mathbf{F}^L$, we define a homomorphism

$$\mathrm{tr}_\beta : \mathrm{End}_{\mathbf{k}}(L, L) \to \mathbf{k}$$

by the formula $\mathrm{tr}_\beta(A) = \mathrm{tr}_L(\beta \cdot A)$, $A \in \mathrm{End}_{\mathbf{k}}(L, L)$. Then tr_β is G-invariant, so that

$$\mathrm{tr}_\beta \circ E_{LL} = \mathrm{tr}_\beta .$$

By a direct computation, we also check that

$$\mathrm{tr}_\beta(\mathrm{Tr}\, v^{\#} \otimes f) = \mathrm{tr}_{\mathbf{F}^L} J\left(v^{\#}(\beta \cdot f)\right).$$

Hence for each $\beta \in \mathbf{F}^L$,

$$\begin{aligned} \mathrm{tr}_\beta\left(m_L E_{LL}(\mathrm{Tr}\, v^{\#} \otimes f)\right) &= m_L \,\mathrm{tr}_{\mathbf{F}^L} J\left(v^{\#}(\beta \cdot f)\right) \\ &= \mathrm{tr}_\beta v^{\#}(f). \end{aligned}$$

This implies that

$$m_L E_{LL}(\mathrm{Tr}\, v^{\#} \otimes f) = v^{\#}(f),$$

because otherwise, letting β be the reciprocal of $m_L E_{LL}(\mathrm{Tr}\, v^{\#} \otimes f) - v^{\#}(f)$ in $\mathbf{F}^L$, we would have $\dim_{\mathbf{k}} L = \mathrm{tr}_L(\mathrm{id}) = 0$, contradicting char $\mathbf{k} = 0$. This completes the proof of Proposition 4.1 in the case (a).

In the case $M = \mathbf{k}[x]$, it follows from the functorial property of the Reynolds operators that $\pi_L(\mathbf{k}[x]_c) \subset \mathbf{k}[x]_c$ for all $c \in \mathbf{N}$. Hence properties (1) and (2) of Proposition 4.1 follow from the finite dimensional case (a). Moreover, if I is an invariant ideal in $\mathbf{k}[x]$, then $I \cap \mathbf{k}[x]_c$ is an invariant subspace of $\mathbf{k}[x]_c$, and

$$I = \bigcup_{c \in \mathbf{N}} I \cap \mathbf{k}[x]_c .$$

Therefore $\pi_L f \in I$ and

$$E_{M^L}(\operatorname{Tr} v_j^{\#},_L \otimes f) \in \operatorname{End}_{\mathbf{k}}(L, I)$$

as required in property (c).

It remains to consider the cases $M = \mathbf{k}[[x]]$, and $M = \mathbf{k}\{x\}$ with $\mathbf{k} = \mathbf{R}, \mathbf{C}$. In each case let $\mathfrak{m}$ be the maximal ideal and let M_c, $c \in \mathbf{N}$, be the invariant subspace of M of polynomials of degree at most c. If $f \in \mathfrak{m}^c$, then $\operatorname{Tr} v^{\#} \otimes f \in \operatorname{End}_{\mathbf{k}}(L, \mathfrak{m}^c)$ for all $v^{\#} \in \operatorname{End}_{\mathbf{F}^L}(L, \mathbf{F}^L)$, so that $\pi_L f \in \mathfrak{m}^c$. Likewise if $f \in M_c$ then $\pi_L f \in M_c$. For each $f \in M$ and $c \in \mathbf{N}$, we write

$$f = T^c f + R^c f ,$$

where $T^c f \in M_c$ and $R^c f \in \mathfrak{m}^{c+1}$. Then for all $f \in M$ and $c \in \mathbf{N}$,

$$\pi_L^2 f - \pi_L f = \pi_L^2(R^c f) - \pi_L(R^c f) \in \mathfrak{m}^{c+1} ,$$

so that $\pi_L^2 = \pi_L$.

For each $c \in \mathbf{N}$, let P_c be the natural projection from M to its subspace of homogeneous polynomials of degree c. Each $f \in M$ may be written $f = \Sigma_c P_c f$. Then $\pi_L \circ P_c = P_c \circ \pi_L$ for every $c \in \mathbf{N}$ and every irreducible G-module L. Suppose that N is a nonzero irreducible G-submodule of $\pi_L(M)$. Then either $P_c(N) = 0$ or $P_c : N \to P_c(N)$ is an equivalence of G-modules. Choose $c \in \mathbf{N}$ such that $P_c(N) \neq 0$. Then N is equivalent to $P_c(N)$ and $P_c(N) = \pi_L(P_c(N)) \subset \pi_L(M_c)$ is equivalent to L, by the finite dimensional case (a). This completes the proof of property (1) for $M = \mathbf{k}[[x]]$ or $\mathbf{k}\{x\}$.

To obtain property (2), we let $N(-1) = \varnothing$ and let $N(c)$, $c \in \mathbf{N}$, be the set of all inequivalent irreducible G-modules appearing in the decomposition of M_c as a direct sum of irreducible G-modules. Then for each $c \in \mathbf{N}$,

$$f - \sum_{L \in N(c)} \pi_L f = R^c f - \sum_{L \in N(c)} \pi_L R^c f \in \mathfrak{m}^{c+1} .$$

Since $\pi_L f = 0$ if $L \notin \cup_c N(c)$, then $\Sigma_L \pi_L f$ converges to f in the Krull topology.

We finally consider property (3) for $M = \mathbf{k}[[x]]$ or $\mathbf{k}\{x\}$. Let I be an invariant ideal in M. Then $I \cap M_c$ is an invariant subspace of M_c. It follows that if $f \in I$, then $\pi_L f \in I + \mathfrak{m}^{c+1}$ for all $c \in \mathbf{N}$, so that $\pi_L f \in I$ by Krull's theorem [14, 16.7]. Moreover

$$\operatorname{End}_{\mathbf{k}}(L, I) = \bigcap_{c \in \mathbf{N}} \operatorname{End}_{\mathbf{k}}(L, I + \mathfrak{m}^{c+1}).$$

Let $f \in I$. Writing $f = T^c f + R^c f$ and using the functorial property of the Reynolds operators, we have

$$E_{M^L}(\operatorname{Tr} v_j^{\#}{}_{,L} \otimes T^c f) \in \operatorname{End}_{\mathbf{k}}(L, I \cap M_c),$$

$$E_{M^L}(\operatorname{Tr} v_j^{\#}{}_{,L} \otimes R^c f) \in \operatorname{End}_{\mathbf{k}}(L, \mathfrak{m}^{c+1})$$

for all $c \in \mathbf{N}$. Since $I + \mathfrak{m}^{c+1} = I \cap M_c + \mathfrak{m}^{c+1}$, it follows that

$$E_{M^L}(\operatorname{Tr} v_j^{\#}{}_{,L} \otimes f) \in \operatorname{End}_{\mathbf{k}}(L, I).$$

This completes the proof of Proposition 4.1.

REFERENCES

[1] Artin, M. Algebraic approximation of structures over complete local rings. *Inst. Hautes Etudes Sci. Publ. Math. 36* (1969), pp. 23-58.

[2] —— Algebraic spaces. *Yale Math. Monographs 3*, Yale University Press, New Haven, Connecticut, 1971.

[3] —— On the solutions of analytic equations. *Inventiones Math. 5* (1968), pp. 277-291.

[4] Becker, J. A counterexample to Artin approximation with respect to subrings. *Math. Ann. 230* (1977), pp. 195-196.

[5] Bierstone, E. General position of equivariant maps. *Trans. Amer. Math. Soc. 234* (1977), pp. 447-466.

[6] Gabrielov, A. M. The formal relations between analytic functions. *Funkcional. Anal. i Prilozen 5* (1971), pp. 64-65 = *Functional Anal. Appl. 5* (1971), pp. 318-319.

[7] —— Formal relations between analytic functions. *Izv. Akad. Nauk SSSR Ser. Mat. 37* (1973), pp. 1056-1090 = *Math. USSR Izvestija 7* (1973), pp. 1056-1088.

[8] Hochschild, G. *The Structure of Lie Groups*. Holden-Day, San Francisco, 1965.

[9] Hochschild, G. and G. D. Mostow. Representations and representative functions of Lie groups. III. *Ann. of Math. 70* (1959), pp. 85-100.

[10] Kirillov, A. *Eléments de la Théorie des Représentations*. Editions Mir, Moscow, 1974.

[11] Luna, D. Fonctions différentiables invariantes sous l'opération d'un groupe réductif. *Ann. Inst. Fourier (Grenoble) 26* (1976), pp. 33-49.

[12] Milman, P. Complex analytic and formal solutions of real analytic equations in $\mathbf{C}^n$ *Math. Ann. 233* (1978), pp. 1-7.

[13] Mumford, D. *Geometric Invariant Theory*. Academic Press, New York, 1965.

[14] Nagata, M. *Local Rings*. Robt. E. Krieger Publ., Huntington, New York, 1975.

[15] Peter, F. und H. Weyl. Die Vollständigkeit der primitiven Darstellungen einer geschlossenen kontinuierlichen Gruppe. *Math. Ann. 97* (1927), pp. 737-755.

[16] Tougeron, J.-C. Solutions d'un système d'équations analytiques réelles et applications. *Ann. Inst. Fourier (Grenoble) 26* (1976), pp. 109-135.

[17] Wavrik, J. J. A theorem on solutions of analytic equations with applications to deformations of complex structures. *Math. Ann. 216* (1975), pp. 127-142.

(Reçu le 29 juin 1978)

Edward Bierstone

Department of Mathematics
University of Toronto
Toronto, Canada, M5S 1A1

Pierre Milman

Department of Mathematics
Purdue University
West Lafayette, Indiana 47907, USA