

# 1. Introduction

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# INVARIANT SOLUTIONS OF ANALYTIC EQUATIONS

by Edward BIERSTONE and Pierre MILMAN

## 1. INTRODUCTION

Let  $\mathbf{k}$  be a field of characteristic zero with a non-trivial valuation.

We consider a system of analytic equations

$$(*) \quad f(x, y) = 0,$$

where

$$f(x, y) = (f_1(x, y), \dots, f_q(x, y))$$

are convergent series in the variables

$$x = (x_1, \dots, x_n),$$

$$y = (y_1, \dots, y_p).$$

Suppose that

$$\bar{y}(x) = (\bar{y}_1(x), \dots, \bar{y}_p(x)), \quad \bar{y}_j(x) \in \mathbf{k}[[x]],$$

are formal power series without constant term which solve (\*); i.e. such that  $f(x, \bar{y}(x)) = 0$ . Let  $c$  be a non-negative integer. Artin's approximation theorem [3] asserts that there exists a convergent series solution

$$y(x) = (y_1(x), \dots, y_p(x)), \quad y_j(x) \in \mathbf{k}\{x\},$$

of (\*), such that

$$y(x) \equiv \bar{y}(x) \pmod{\mathfrak{m}^c}.$$

Here  $\mathfrak{m}$  denotes the maximal ideal of  $\mathbf{k}[[x]]$ .

Artin also proved an algebraic analogue of this theorem [1]. It says that if  $f(x, y) = 0$  is a system of polynomial equations with formal series solution  $\bar{y}(x)$ , then a series solution  $y(x)$  may be found such that the  $y_j(x)$  are algebraically dependent on  $x_1, \dots, x_n$  (we will say that the  $y_j(x)$  are "algebraic"; cf. [2]). In this analogue  $\mathbf{k}$  is an arbitrary field.

Let  $G$  be a reductive algebraic group (i.e.  $G$  is linear and every rational representation of  $G$  is completely reducible). Suppose that  $G$  acts linearly on  $V = \mathbf{k}^n$  and  $W = \mathbf{k}^p$ . We will say that  $\bar{y}(x) \in \mathbf{k}[[x]]^p$  is *equivariant* if

$$\bar{y}(gx) = g\bar{y}(x), \quad g \in G.$$

We will prove the following theorem.

**THEOREM A.** Suppose  $\mathbf{k} = \mathbf{R}$  or  $\mathbf{C}$ , and that  $\bar{y}(x) \in \mathbf{k}[[x]]^p$  is an equivariant formal power series solution of (\*),  $\bar{y}(0) = 0$ . Let  $c \in \mathbf{N}$ . Then there exists an equivariant convergent series solution  $y(x)$  of (\*), such that  $y(x) \equiv \bar{y}(x) \pmod{\mathfrak{m}^c}$ .

Moreover, if  $f(x, y) = 0$  is a system of polynomial equations (where  $\mathbf{k}$  is any field), then there exists an equivariant algebraic solution  $y(x)$ , such that  $y(x) \equiv \bar{y}(x) \pmod{\mathfrak{m}^c}$ .

*Remark 1.1.* Theorem A may be regarded in the context of the question: What properties of a formal solution of (\*) may be preserved in an analytic solution? Artin [2] asked whether there is a convergent solution such that some of the variables  $x_i$  are missing in some of the series  $y_j(x)$ , provided there is a formal solution with the same property. Gabrielov [6] answered this question negatively (see also [4]). In [12] it is shown that if a formal solution of a system of real analytic equations satisfies the Cauchy-Riemann equations, then it may be approximated by complex analytic solutions.

*Remark 1.2.* Suppose that  $\pi(x) \in \mathbf{C}\{x\}^r$  is an analytically regular germ of an analytic mapping (terminology of Gabrielov [7]). Let  $F_i(x) \in \mathbf{C}\{x\}^p$ ,  $i = 1, \dots, q$ . We may ask whether formal relations among the  $F_i$  of the form  $(h_1(\pi(x)), \dots, h_q(\pi(x)))$ ; i.e.  $q$ -tuples of formal power series of this form such that

$$\sum_{i=1}^q h_i(\pi(x)) F_i(x) = 0,$$

are generated by analytic relations of the same form. This question generalizes Gabrielov's problem in [7]. The answer is *no* in general, but the method of our proof of Theorem A shows it is *yes* if  $\pi$  is a finite analytic germ. As in our proof of Theorem A, it is then easy to see that a formal solution  $\bar{y}(\pi(x))$  of a system of complex analytic equations  $f(x, y) = 0$  may be approximated by analytic solutions of the same form. We are grateful to Joseph Becker for pointing out the latter result to us.

*Remark 1.3.* Tougeron [16] has proved a generalization of Artin's theorem which asserts, in particular, that every formal solution  $\bar{y}(x)$  of (\*) such that  $\bar{y}(0) = 0$  is the formal Taylor series at 0 of an infinitely differentiable solution. The proof of Theorem A also gives an equivariant version of Tougeron's theorem.

Theorem A is closely related to the second result of this paper.

**THEOREM B.** *Suppose that  $G$  acts linearly on  $V = \mathbf{k}^n$ , and that  $X$  is a closed algebraic subset of  $V$  which is invariant under the action of  $G$ . Then there exists a linear action of  $G$  on a finite dimensional vector space  $Y = \mathbf{k}^q$ , and an equivariant polynomial mapping  $F: V \rightarrow Y$  such that  $X = F^{-1}(0)$ .*

*If  $\mathbf{k} = \mathbf{R}$  or  $\mathbf{C}$ , and  $X$  is a germ at 0 of a closed analytic subset of  $V$  which is invariant under the action of  $G$ , then there exists a vector space  $Y = \mathbf{k}^q$  on which  $G$  acts linearly, and a germ  $F$  of an equivariant analytic mapping of some neighborhood of  $0 \in V$  into  $Y$ , such that  $X = F^{-1}(0)$ .*

A linear action of  $G$  on  $\mathbf{k}^n$  induces an action on  $\mathbf{k}[[x]] = \mathbf{k}[[x_1, \dots, x_n]]$  (respectively  $\mathbf{k}\{x\}$ ,  $\mathbf{k}[x]$ ) such that

$$(g \cdot f)(x) = f(g^{-1}x)$$

for all  $g \in G$  and  $f(x) \in \mathbf{k}[[x]]$  (respectively  $\mathbf{k}\{x\}$ ,  $\mathbf{k}[x]$ ). Let  $\mathbf{k}[[x]]^G$  (respectively  $\mathbf{k}\{x\}^G$ ,  $\mathbf{k}[x]^G$ ) be the subset of elements fixed by  $G$  (the invariant elements).

*Remark 1.4.* It is well-known that if  $\mathbf{k} = \mathbf{R}$  and  $G$  is compact, then the conclusion of Theorem B holds with  $F \in (\mathbf{R}[x]^G)^q$  (or  $F \in (\mathbf{R}\{x\}^G)^q$  in the analytic case). In general, invariants separate only disjoint Zariski closed invariant subsets of  $\mathbf{k}^n$ , so that invariant closed algebraic or analytic subsets needn't be defined by invariant equations.

We will prove Theorem B in the following section, considering separately the complex analytic, real analytic, and algebraic cases. These results may also be obtained in a unified way, at least in characteristic zero, from an explicit projection formula related to the Fourier transform (cf. [15], [10, 12.2]). This formula may be of independent interest, and we have included it in section 4. In section 3 we will deduce Theorem A from Theorem B.

The authors enjoyed several conversations with Joseph Becker on the results in this paper.