3. Exceptional sets

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Characterization A2. The singularity of $f^{-1}(0)$ is rational.

Characterizations A1 and A2 will both be shown equivalent to Characterization A3.

3. Exceptional sets

Let V be as above, and let $\pi: M \to V$ be a resolution of V. The *exceptional set* $E = \pi^{-1}(\mathbf{v})$ is compact, one-dimensional, and connected, and hence is a union of irreducible complex curves $E_1, ..., E_s$. It is possible to arrange that the E_i are non-singular, the intersection of E_i and E_j is transverse for $i \neq j$, and no three E_i meet at a point. Such a resolution is called *good.* If, in addition, the intersection of E_i and E_j is empty or one point, the resolution is *very good*; this is possible to arrange as well.

Suppose that the resolution is good. Let $E_i \cdot E_j$ equal the number of points of intersection of E_i and E_j if $i \neq j$ (always a non-negative integer), or the first Chern class of the normal bundle to E_i evaluated on the orientation class of E_i if $i = j$ (the self-intersection of E_i). The matrix $\{E_i \cdot E_j\}$ is called the *intersection matrix of the resolution.* It is proved in [Du Val 2] (see also [Mumford; Laufer 1, p. 49]) that this matrix is negative definite. Conversely, given a collection of curves $E = E_1 \cup ... \cup E_s$ in a two-dimensional manifold M with negative definite intersection matrix $\{E_i \cdot E_j\}$, a theorem of Grauert says that the quotient space M/E has a normal complex structure and that the projection map $M \to M/E$ is analytic [Laufer 1, p. 60].

Characterization A3. The minimal resolution of $f^{-1}(0)$ is very good, and its exceptional set consists of curves of genus 0 and self-intersection -2.

The equivalence of Characterizations A2 and A3 is proved in [Du Val 1], and [Artin]. The following facts are needed:

(i) Let $M \to V$ be a resolution of a normal singularity V as above. There is a certain unique non-zero divisor $Z = \Sigma n_i E_i$ on M with $n_i \geqslant 0$ called the *fundamental cycle,* and it is shown that the singularity of V is rational if and only if the analytic Euler characteristic $\chi(Z)$ of Z is 1 (that is, the arithmetic genus of Z is 0) [Artin, Theorem 3]. It is easy to see that the support of Z is the whole exceptional set of E.

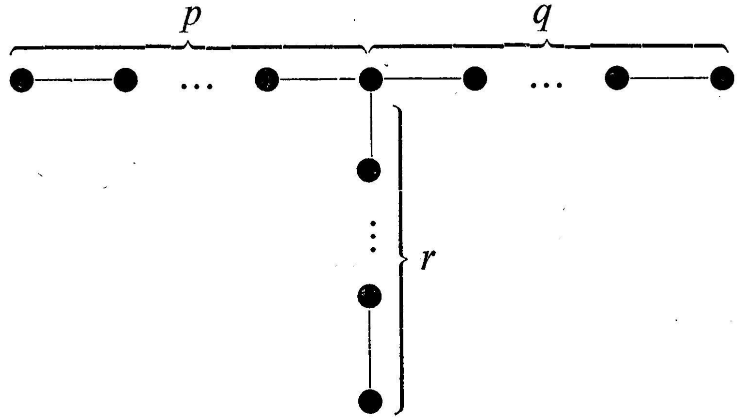
(ii) Any resolution of a rational singularity V is very good, and the curves in the exceptional set are of genus zero [Brieskorn 2, Lemma 1.3].

(iii) A rational singularity V embeds in codimension one if and only if it is a double point, which is true if and only if $Z^2 = -2$ [Artin, Corollary 6].

(A2) ⇒ (A3): We only need show $E_i^2 = -2$ for all i. Certainly $E_i^2 \leqslant -2$, since if $E_i^2 = -1$ the resolution could be contracted by Castelnuovo's criterion, and $E_i^2 \geqslant 0$ would contradict the fact that the matrix $\{E_i \cdot E_j\}$ is negative definite. Let K be the canonical class of M. (This exists since V is Gorenstein; see for instance [Durfee 2].) The adjunction formula $-E_i \cdot K = E_i^2 + 2$ then shows that $E_i \cdot K \geqslant 0$ for each i. The Riemann-Roch Theorem $\chi(Z) = -\frac{1}{2}(Z^2 + Z \cdot K)$ implies that $Z \cdot K = 0$. Thus $0 = Z \cdot K \geqslant (E_1 + \ldots + E_s) \cdot K \geqslant E_i \cdot K \geqslant 0$. Hence $E_i \cdot K = 0$ for all i, so again by the adjunction formula, $E_i^2 = -2$.

(A3) ⇒ (A2): The adjunction formula implies that $E_i \cdot K = 0$ for all i; since the matrix $\{E_i \cdot E_j\}$ is negative definite, $K = 0$. Thus $\chi(Z) = \frac{1}{2} Z^2$ by the Riemann-Roch Theorem. Since $\chi(Z) \leqslant 1$ and $Z^2 < 0$ (again since $\{E_i \cdot E_j\}$ is negative definite), $\chi(Z)$ must be 1 and Z^2 must be -2. This completes the proof.

Now, exactly what exceptional sets satisfy Characterization A3? First some algebra. It is possible to associate a weighted graph to any symmetric integral bilinear form $\langle\, ,\rangle$ on a free module with basis $e_1, \ldots, e_s$ satisfying $\langle e_i, e_j \rangle \geqslant 0$ for $i \neq j$: The vertices of the graph are $v_1, \ldots, v_s$, two vertices v_i and v_j are joined by $\langle e_i, e_j \rangle$ edges, and the vertex v_i is weighted by the integer $\langle e_i, e_i \rangle$. Conversely, a weighted graph defines such a bilinear form. Let $T_{p,q,r}$ be the weighted graph

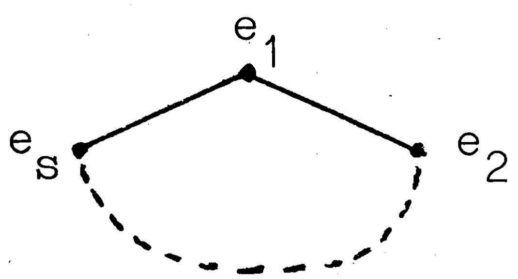


where p, q, and r are positive integers, and all vertices are weighted by -2.

Lemma 3.1 [Hirzebruch 2, p. 217]. The only connected graphs weighted by -2 and whose associated bilinear form is negative definite are of type $T_{p,q,r}$, where p, q, and r are positive integers satisfying $p^{-1} + q^{-1} + r^{-1} > 1$.

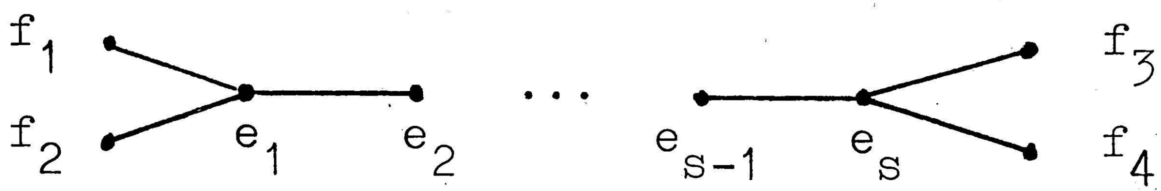
Proof. (a) If the bilinear form associated to a graph is negative definite, so is the bilinear form associated to any subgraph.

(b) The graph $(s \geqslant 2)$



where all vertices $e_1, \ldots, e_s$ are weighted by -2, is not negative definite, since $(e_1 + \ldots + e_s)^2 = 0$.

(c) The graph



where all vertices are weighted by -2, is not negative definite, since $(2e_1 + \ldots + 2e_s + f_1 + \ldots + f_4)^2 = 0$.

Thus the graph must be of the form $T_{p,q,r}$. An elementary argument shows that the bilinear form of $T_{p,q,r}$ is isomorphic over the rationals to the direct sum of a negative definite form and the one-dimensional form $\langle 1 - p^{-1} - q^{-1} - r^{-1} \rangle$. Hence $T_{p,q,r}$ is negative definite if and only if $p^{-1} + q^{-1} + r^{-1} > 1$. This proves the lemma.

The only triples of positive integers (p, q, r) satisfying $p^{-1} + q^{-1} + r^{-1} > 1$ are of course just $(1, 1, r)$ for $r \geqslant 1$, $(2, 2, r)$ for $r \geqslant 2$, $(2, 3, 3)$, $(2, 3, 4)$, and $(2, 3, 5)$.

The *dual graph of a resolution of a singularity* is defined to be the weighted graph associated to the intersection matrix of the resolution. Applying the above facts, we see that Characterization A3 is equivalent to:

Characterization A3′. The minimal resolution of $f^{-1}(0)$ is listed in column (3) of Table 1.

Next we show that Characterization A1 and A3 are equivalent. Characterization A1 implies Characterization A3 since the singularities of the functions f listed in column 1 of Table 1 have minimal resolutions as in column 3. (I believe that this first appeared in [Hirzebruch 1].) The converse follows since the singularities listed are taut [Brieskorn 2; Tjurina 3; Laufer 4]. (Two resolutions $\pi: M \to V$ and $\pi': M' \to V'$ are *topologically equivalent* if their exceptional sets are homeomorphic by a homeomorphism preserving the self-intersection numbers. A singularity V is *taut* if any other singularity with a good resolution topologically equivalent to a good resolution of V is then isomorphic to V.)

The classification of rational double points has been generalized in several ways: to rational triple points [Artin, p. 135], to elliptic singularities [Wagreich 1], and to minimally elliptic singularities [Laufer 5]. The Dynkin diagrams B_n, C_n, F_4 and G_2 occur when resolving singularities over non-algebraically closed fields [Lipman 1]. There is also a relation with simple complex Lie groups [Brieskorn 3].

4. Absolutely isolated double points

There are at least three methods of resolving the singularity of the germ of a normal two-dimensional complex space V. The first method is one of local uniformization; this is originally due to Jung, and is described in detail in [Laufer 1]. The second method, due to Zariski, is to alternately blow up points and normalize. The third method (which generalizes to higher dimensions), is to blow up points and non-singular curves.

The singularity of V is *absolutely isolated* if it may be resolved by blowing up points alone, that is, it is not necessary to normalize or blow up curves. For example, the singularity of the zero locus of $f(x, y, z) = x^k + y^k + z^k$ is absolutely isolated, since it may be resolved by blowing up the origin once.

The singularity of V is a *double point* if its local ring is of multiplicity two. If V is $f^{-1}(0)$, this is equivalent to the lowest non-zero homogeneous term in the power series expansion of f being quadratic.

Characterization A4. The singularity of $f^{-1}(0)$ is an absolutely isolated double point.

The equivalence of Characterizations A1 and A4 was proved directly in [Kirby]. Later, it was shown [Tjurina 2; Lipman 1] that all rational singularities are absolutely isolated (thus showing Characterization A2 implies A4), and in [Brieskorn 1, Satz 1] that A4 implies A3.