5. Quotient singularities

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5. Quotient Singularities

Let U be a neighborhood of the origin $\mathbf{0}$ in $\mathbf{C}^2$ and let H be a finite group of analytic automorphisms of U fixing $\mathbf{0}$. The quotient space U/H has the structure of a normal two-dimensional complex analytic space with an isolated singularity, and the projection map $U \to U/H$ is analytic [Cartan]. An analytic space V is called a *quotient singularity* if there is a U and H as above such that V is isomorphic to U/H.

An important example of a quotient singularity is $\mathbf{C}^2/G$, where G is some finite subgroup of $GL(2, \mathbf{C})$. The space $\mathbf{C}^2/G$ is not just analytic, but algebraic. For any finite subgroup G of $GL(2, \mathbf{C})$, the ring of functions on the algebraic variety $\mathbf{C}^2/G$ is isomorphic to the subring of invariant polynomials in $GL(2, \mathbf{C})$. Hence to find $\mathbf{C}^2/G$ it suffices to find this subring of invariant polynomials. Note that a finite subgroup G of $GL(2, \mathbf{C})$ or $SL(2, \mathbf{C})$ is conjugate to a finite subgroup of $U(2)$ or $SU(2)$ respectively, since it is possible to choose an invariant Hermitian metric on $\mathbf{C}^2$. A subgroup $G \subset GL(2, \mathbf{C})$ is *small* if no $g \in G$ has 1 as an eigenvalue of multiplicity one. [Prill, p. 380].

Proposition 5.1. *Let V be the germ of a normal two-dimensional complex analytic space. The following statements are equivalent.*

(a) *V is a quotient singularity.*

(b) *V is isomorphic to $\mathbf{C}^2/G$, for some finite subgroup G of $GL(2, \mathbf{C})$.*

(c) *V is isomorphic to $\mathbf{C}^2/G$, for some small finite subgroup of $GL(2, \mathbf{C})$.*

Condition (a) implies condition (b) by the usual linearization argument [Brieskorn 2, Lemma 2.2]. It is shown in [Prill, p. 380] that condition (b) implies condition (c). Obviously (c) implies (a). The following theorem is also proved in [Prill]: Let G and G' be small finite subgroups of $GL(2, \mathbf{C})$. Then the analytic spaces $\mathbf{C}^2/G$ and $\mathbf{C}^2/G'$ are isomorphic if and only if G and G' are conjugate.

Characterization A5. The analytic space $f^{-1}(0)$ is a quotient singularity.

Since quotient singularities are rational [Brieskorn 2, p. 340], Characterization A5 implies Characterization A2. The converse will follow in round-about fashion.

Consider $SU(2)$, which is of course isomorphic to the group S^3 of unit quaternions. The finite subgroups of S^3 are the cyclic group and the inverse images of the finite subgroups of the rotation group $SO(3)$ under the double cover $S^3 \to SO(3)$; these groups are listed in column 5 of Table 1.

PROPOSITION 5.2. *Let G be a non-trivial finite subgroup of $SU(2)$ as listed in column 5 of Table* 1. *Then $\mathbf{C}^2/G$ is isomorphic to $f^{-1}(0)$, where f is the corresponding polynomial in column 1.*

In particular, for each polynomial f in column 1 of Table 1 the analytic space $f^{-1}(0)$ is isomorphic to a quotient singularity. This proposition is proved by classical invariant theory. For the cyclic group it is easy: Let $G \subset SU(2)$ be the cyclic group of order k, generated by the transformation $(u, v) \to (\eta u, \eta^{-1} v)$ where η is a primitive k-th root of unity. Then we claim that $\mathbf{C}^2/G$ is isomorphic to

$$V = \{(x, y, z) \in \mathbf{C}^3 : x^k = yz\}.$$

Let $p_1(u, v) = uv$, $p_2(u, v) = u^k$, $p_3(u, v) = v^k$, and let $p = (p_1, p_2, p_3)$ define a map of $\mathbf{C}^2$ to $\mathbf{C}^3$. The image of p is exactly V. Since $p_i(gu, gv) = p_i(u, v)$ for all g in G, the map p induces a map $\bar{p}$ of $\mathbf{C}^2/G$ to V. The map $\bar{p}$ is easily seen to be injective, and thus is an isomorphism, since $\mathbf{C}^2/G$ and V are normal.

The proof for the other finite subgroups G of S^3 is similar, and may be found in [Du Val 3]: The elements of G are listed, the subring R of $\mathbf{C}[u, v]$ of invariant polynomials is found to be generated by three homogeneous polynomials p_1, p_2, p_3 of various degrees, and they satisfy exactly one weighted homogeneous relation $f(p_1, p_2, p_3) = 0$. It follows that $\mathbf{C}^2/G$ is isomorphic to the zero locus of f. Special cases of this proof go back to [Klein]. It is also possible to give a simpler uniform proof using vertices, edges, and faces when G is the commutator subgroup $[H, H]$ of another finite subgroup H of S^3 [Milnor 2, §4].

[Du Val 3, §30] gives a geometric description of the links of these singularities as regular solids with opposite faces identified. (The *link* of a germ $V \subset \mathbf{C}^n$ at $\mathbf{v}$ is V intersected with a suitably small sphere about $\mathbf{v}$.)

The finite subgroups of $GL(2, \mathbf{C})$ are listed in [Du Val 3, §21] and the corresponding quotient singularities are studied in [Brieskorn 2, p. 348]. The ring of invariant polynomials has been computed for the cyclic and dihedral subgroups [Riemenschneider 1,2]. Generalizations of quotient singularities and their relation to weighted homogeneous polynomials may be found in [Milnor 2; Dolgachev].

Characterization A5′. The analytic space $f^{-1}(0)$ is isomorphic to $\mathbf{C}^2/G$, where G is a finite subgroup of $SU(2)$.

Proposition 5.2 shows that characterizations A5′ and A1 are equivalent. Clearly Characterization A5′ implies A5; since A5 implies A2, they are all equivalent.

COROLLARY 5.3. *Let G be a small finite subgroup of $GL(2, \mathbf{C})$. Then $G \subset SL(2, \mathbf{C})$ if and only if $\mathbf{C}^2/G$ embeds in codimension one.*

This corollary follows from the above case-by-case analysis. J. Wahl points out that it is also possible to prove it directly, using the following two facts:

Fact 1. Let G be a small finite subgroup of $GL(2, \mathbf{C})$. Then $G \subset SL(2, \mathbf{C})$ if and only if the singularity of $\mathbf{C}^2/G$ is Gorenstein.

This is a special case of [Watanabe]. A germ of a normal two-dimensional complex space is *Gorenstein* if there is a nowhere-vanishing holomorphic two-form on its regular points.

Fact 2. Let V be the germ at $\mathbf{v}$ of a two-dimensional rational singularity. Then V is Gorenstein if and only if V embeds in codimension 1.

Proof. Any singularity embedded in codimension one is Gorenstein. Conversely, suppose V is Gorenstein. Let $\pi: M \to V$ be the minimal resolution of V, and let $E_1 \cup \ldots \cup E_s = \pi^{-1}(v)$ be its exceptional set as in Section 3. Since V is Gorenstein, there is a divisor K on M (the *canonical class*) satisfying the adjunction formula. Furthermore $K \cdot E_i \geqslant 0$ for all i since the resolution is minimal, so $K \leqslant 0$ [Artin, bottom of p. 130]. If $K < 0$, then $-K > 0$, so arithmetic genus p of $-K$ satisfies $p(-K) \leqslant 0$ [Artin, Proposition 1]. On the other hand, $p(-K) = 1 - \chi(-K) = 1$ by the Riemann-Roch Theorem, a contradiction. Hence $K = 0$. Thus $K \cdot E_i = 0$ for all i, so V is a double point and embeds in codimension one, as in the proof that Characterization A3 implies Characterization A2.

6. THE LOCAL FUNDAMENTAL GROUP

Let V be the germ of a normal two-dimensional complex analytic space with an isolated singularity at $\mathbf{v}$. Without loss of generality, we may assume that V is a *good neighborhood* of $\mathbf{v}$, that is, that there is a neighborhood basis V_i of $\mathbf{v}$ in V such that each $V_i - \mathbf{v}$ is a deformation retract of $V - \mathbf{v}$ [Prill]. The *local fundamental group* of V at $\mathbf{v}$ is then defined as $\pi_1(V - \mathbf{v})$. This group is trivial if and only if V is nonsingular at $\mathbf{v}$ [Mumford].