6. The local fundamental group

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Proposition 5.2 shows that characterizations A5′ and A1 are equivalent. Clearly Characterization A5′ implies A5; since A5 implies A2, they are all equivalent.

COROLLARY 5.3. *Let G be a small finite subgroup of $GL(2, \mathbf{C})$. Then $G \subset SL(2, \mathbf{C})$ if and only if $\mathbf{C}^2/G$ embeds in codimension one.*

This corollary follows from the above case-by-case analysis. J. Wahl points out that it is also possible to prove it directly, using the following two facts:

Fact 1. Let G be a small finite subgroup of $GL(2, \mathbf{C})$. Then $G \subset SL(2, \mathbf{C})$ if and only if the singularity of $\mathbf{C}^2/G$ is Gorenstein.

This is a special case of [Watanabe]. A germ of a normal two-dimensional complex space is *Gorenstein* if there is a nowhere-vanishing holomorphic two-form on its regular points.

Fact 2. Let V be the germ at $\mathbf{v}$ of a two-dimensional rational singularity. Then V is Gorenstein if and only if V embeds in codimension 1.

Proof. Any singularity embedded in codimension one is Gorenstein. Conversely, suppose V is Gorenstein. Let $\pi: M \to V$ be the minimal resolution of V, and let $E_1 \cup \ldots \cup E_s = \pi^{-1}(v)$ be its exceptional set as in Section 3. Since V is Gorenstein, there is a divisor K on M (the *canonical class*) satisfying the adjunction formula. Furthermore $K \cdot E_i \geqslant 0$ for all i since the resolution is minimal, so $K \leqslant 0$ [Artin, bottom of p. 130]. If $K < 0$, then $-K > 0$, so arithmetic genus p of $-K$ satisfies $p(-K) \leqslant 0$ [Artin, Proposition 1]. On the other hand, $p(-K) = 1 - \chi(-K) = 1$ by the Riemann-Roch Theorem, a contradiction. Hence $K = 0$. Thus $K \cdot E_i = 0$ for all i, so V is a double point and embeds in codimension one, as in the proof that Characterization A3 implies Characterization A2.

6. THE LOCAL FUNDAMENTAL GROUP

Let V be the germ of a normal two-dimensional complex analytic space with an isolated singularity at $\mathbf{v}$. Without loss of generality, we may assume that V is a *good neighborhood* of $\mathbf{v}$, that is, that there is a neighborhood basis V_i of $\mathbf{v}$ in V such that each $V_i - \mathbf{v}$ is a deformation retract of $V - \mathbf{v}$ [Prill]. The *local fundamental group* of V at $\mathbf{v}$ is then defined as $\pi_1(V - \mathbf{v})$. This group is trivial if and only if V is nonsingular at $\mathbf{v}$ [Mumford].

Proposition 5.1 (bis). *The following statement is equivalent to those listed above.*

(d) *The local fundamental group of* V *is finite.*

It is shown in [Prill, p. 381; Brieskorn 2, p. 344] that conditions (a) and (d) are equivalent.

Characterization A6. The local fundamental group of $f^{-1}(0)$ is finite.

Thus Characterizations A5 and A6 are equivalent.

There is an algorithm for computing the local fundamental group of V from a resolution [Mumford], and singularities V with finite, nilpotent and solvable local fundamental group have been classified [Brieskorn 2; Wagreich 2]. When V is a complete intersection, this classification is particularly simple [Durfee 2, Proposition 3.3].

7. Volume

Let $f(x, y, z)$ be the germ at the origin $\mathbf{0}$ of a complex analytic function, and suppose that $f(\mathbf{0}) = 0$ and that the origin is an isolated critical point of f. There is an $\varepsilon > 0$ such that $f^{-1}(\mathbf{0})$ intersects all spheres of radius ε' about $\mathbf{0}$ transversally for $0 < \varepsilon' \leqslant \varepsilon$. (See Section 12.) For $t \in \mathbf{C}$, let

$$V_t = f^{-1}(t) \cap D_\varepsilon^6$$

where D_ε^6 is the closed disk of radius ε about $\mathbf{0}$. The function $f(x, y, z)$ takes the constant value t on V_t, so $\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz \equiv 0$ there. Hence a nowhere-vanishing holomorphic two-form ω_t on V_t may be defined by the equivalent expressions

$$\omega_t = \frac{dy \wedge dz}{\partial f / \partial x} = \frac{dz \wedge dx}{\partial f / \partial y} = \frac{dx \wedge dy}{\partial f / \partial z},$$

Characterization A7. The integral $\int_{V_0} \omega_0 \wedge \bar{\omega}_0$ is finite.

Note that the form $\omega_0 \wedge \bar{\omega}_0$ takes positive real values. The equivalence of Characterizations A2 and A7 is due to Laufer, and follows easily from his expression for the geometric genus in terms of forms [Laufer 2, Corollary 3.6].

A different formulation of this characterization is due to E. Looijenga (unpublished): Let $\Delta(r) = \{t \in \mathbf{C}: t < r\}$, let

$$X(r) = f^{-1}(\Delta(r)) \cap D_\varepsilon^6$$

and let $\mathrm{vol}(X(r))$ be its volume in $\mathbf{C}^3$.