9. CHARACTERIZATIONS UNDER RIGHT AND CONTACT EQUIVALENCE

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The fundamental lemmas used in the classification are as follows:

Lemma 8.2. If $\mathfrak{m}^{k+1} \subset \mathfrak{m}^2 \Delta f$ then f is k-determined.

For the proof, see [Arnold 1, Lemma 3.2; Zeeman, Theorem 2.9; Siersma, p. 8]. Note that $\mathfrak{m}^{k-1} \subset \Delta f$ implies that $\mathfrak{m}^{k+1} \subset \mathfrak{m}^2 \Delta f$. The *corank* of f is defined as $n + 1$ minus the rank of the Hessian matrix $\{(\partial^2 f/\partial z_i \partial z_j)(\mathbf{0})\}$. The proof of part (a) of the following lemma may be found in [Arnold 1, Lemma 4.1; Siersma Lemma 3.2].

Splitting Lemma 8.3. (a) Let $f(z_0, ..., z_n) \in \mathscr{F}$ be of corank $r + 1$. Then there is a $g(z_0, ..., z_r) \in \mathfrak{m}^3$ such that

$$f(z_0, \ldots, z_n) \sim g(z_0, \ldots, z_r) + z_{r+1}^2 + \ldots + z_n^2 .$$

(b) Let $g(z_0, ..., z_r)$ and $g'(z_0, ..., z_r) \in \mathscr{F} \cap \mathfrak{m}^3$. If

$$g(z_0, \ldots, z_r) + z_{r+1}^2 + \ldots + z_n^2 \sim g'(z_0, \ldots, z_r) + z_{r+1}^2 + \ldots + z_n^2$$

then

$$g(z_0, \ldots, z_r) \sim g'(z_0, \ldots, z_r) .$$

The classification proceeds by low corank and low Milnor number. A germ of corank 0 is right equivalent to $z_0^2 + \ldots + z_n^2$, a germ of corank 1 and Milnor number $k > 1$ is right equivalent to $z_0^{k+1} + z_1^2 + \ldots + z_n^2$, and so forth. The proofs are not hard [Arnold 1, Zeeman, Siersma]. Table 2, for instance, includes all right-equivalence classes of germs with Milnor number $\mu \leqslant 9$.

9. Characterizations under right and contact equivalence

Characterization B1. The germ f is right equivalent to one of the germs in Table 2a.

Characterization B2. The germ f is contact equivalent to one of the germs in Table 2a.

When $n = 2$, Characterization B2 is the same as Characterization A1. Clearly Characterization B1 implies Characterization B2. Since all the germs in Table 2a are weighted homogeneous (§16), the converse follows from the next lemma.

Lemma 9.1. Let g be a weighted homogeneous polynomial, and suppose that a germ $f \in \mathscr{F}$ is contact equivalent to g. Then f is right equivalent to g.

Proof. To say that f is contact equivalent to g means that there is a germ of an analytic isomorphism $h: (\mathbf{C}^{n+1}, \mathbf{0}) \to (\mathbf{C}^{n+1}, \mathbf{0})$ and a function $u: \mathbf{C}^{n+1} \to \mathbf{C}$ with $u(\mathbf{0}) \neq 0$ such that $f = u \cdot (g \circ h)$. Let $h = (h^0, ..., h^n)$ be the components of h, and suppose that g is weighted homogeneous with weights $(w_0, ..., w_n)$. Then,

$$\begin{aligned} f(z_0, ..., z_n) &= u(z_0, ..., z_n) \cdot g\left(h^0(z_0, ..., z_n), ..., h^n(z_0, ..., z_n)\right) \\ &= g\big((u(z_0, ..., z_n))^{1/w_0}\ h^0(z_0, ..., z_n), ..., \\ &\qquad (u(z_0, ..., z_n))^{1/w_n}\ h^n(z_0, ..., z_n)\big). \end{aligned}$$

Hence f is right equivalent to g.

10. Degeneration

Let J_k be the set of k-jets of germs in $\mathcal{O}$. There is a projection of $\mathcal{O}$ to J_k by mapping germs to their power series expansion truncated in degree k. The ring $\mathcal{O}$ becomes a topological space by letting a basis of open sets be inverse images of open sets in J_k, for all k.

The group of germs of analytic automorphisms fixing $\mathbf{0}$ acts on $\mathcal{O}$, and the orbits of this action *(right-equivalence orbits)* are the right-equivalence classes. Similarly, there is a contact equivalence group which acts on $\mathcal{O}$, and the orbits of this action *(contact-equivalence orbits)* are the contact equivalence classes [Mather, §2]. A right-equivalence orbit is always contained in a contact-equivalence orbit; Lemma 9.1 says that the right-equivalence orbit of a germ f in Table 2a or b equals its contact-equivalence orbit.

A subset A of $\mathcal{O}$ is said to *right* (or *contact*) *degenerate* to a subset B of $\mathcal{O}$ if the closure of the right (or contact) equivalence orbit of A contains B. If A degenerates to B, then B *simplifies* to A (written $A \leftarrow B$). Right degeneracy is also called *adjacency*. For example, when $n = 0$, the germ z_0^k right degenerates to the germ z_0^l for $k < l$, since the one-parameter family $tz_0^k + (1-t)\, z_0^l$ is z_0^l when $t = 0$, and is right-equivalent to z_0^k when $t \neq 0$. All germs of low codimension can be arranged according to right degeneracy in fascinating tables [Arnold 3; Siersma]. Table 3 lists some (but not all) of the simplifications that occur. The following proposition is a principal consequence of the work on degeneration.

Proposition 10.1.

(i) *The germs in Table 2a right simplify only to each other.*

(ii) *The germs in Table 2b right simplify only to the germs in Table 2a.*