10. Degeneration

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Proof. To say that f is contact equivalent to g means that there is a germ of an analytic isomorphism $h: (\mathbf{C}^{n+1}, \mathbf{0}) \to (\mathbf{C}^{n+1}, \mathbf{0})$ and a function $u: \mathbf{C}^{n+1} \to \mathbf{C}$ with $u(\mathbf{0}) \neq 0$ such that $f = u \cdot (g \circ h)$. Let $h = (h^0, ..., h^n)$ be the components of h, and suppose that g is weighted homogeneous with weights $(w_0, ..., w_n)$. Then,

$$\begin{aligned} f(z_0, ..., z_n) &= u(z_0, ..., z_n) \cdot g\left(h^0(z_0, ..., z_n), ..., h^n(z_0, ..., z_n)\right) \\ &= g\left((u(z_0, ..., z_n))^{1/w_0}\, h^0(z_0, ..., z_n), ..., \right. \\ &\qquad \left. (u(z_0, ..., z_n))^{1/w_n}\, h^n(z_0, ..., z_n)\right). \end{aligned}$$

Hence f is right equivalent to g.

10. Degeneration

Let J_k be the set of k-jets of germs in $\mathcal{O}$. There is a projection of $\mathcal{O}$ to J_k by mapping germs to their power series expansion truncated in degree k. The ring $\mathcal{O}$ becomes a topological space by letting a basis of open sets be inverse images of open sets in J_k, for all k.

The group of germs of analytic automorphisms fixing $\mathbf{0}$ acts on $\mathcal{O}$, and the orbits of this action *(right-equivalence orbits)* are the right-equivalence classes. Similarly, there is a contact equivalence group which acts on $\mathcal{O}$, and the orbits of this action *(contact-equivalence orbits)* are the contact equivalence classes [Mather, §2]. A right-equivalence orbit is always contained in a contact-equivalence orbit; Lemma 9.1 says that the right-equivalence orbit of a germ f in Table 2a or b equals its contact-equivalence orbit.

A subset A of $\mathcal{O}$ is said to *right* (or *contact*) *degenerate* to a subset B of $\mathcal{O}$ if the closure of the right (or contact) equivalence orbit of A contains B. If A degenerates to B, then B *simplifies* to A (written $A \leftarrow B$). Right degeneracy is also called *adjacency*. For example, when $n = 0$, the germ z_0^k right degenerates to the germ z_0^l for $k < l$, since the one-parameter family $tz_0^k + (1-t)\, z_0^l$ is z_0^l when $t = 0$, and is right-equivalent to z_0^k when $t \neq 0$. All germs of low codimension can be arranged according to right degeneracy in fascinating tables [Arnold 3; Siersma]. Table 3 lists some (but not all) of the simplifications that occur. The following proposition is a principal consequence of the work on degeneration.

Proposition 10.1.

(i) *The germs in Table 2a right simplify only to each other.*

(ii) *The germs in Table 2b right simplify only to the germs in Table 2a.*

(iii) *The germs in Table 2c right simplify only to the germs in Table 2b and 2a.*

(iv) *A germ in $\mathscr{F}$ not right equivalent to a germ in Table 2a, b, or c right simplifies to a germ in Table 2c.*

11. Simple germs and moduli

A germ $f \in \mathfrak{m}$ is said to be *right* (or *contact*) *simple* if there is a neighborhood of f in $\mathfrak{m}$ intersecting only finitely many right (or contact) equivalence orbits. In the language of algebraic geometry, a germ f is contact simple if and only if the versal deformation of $f^{-1}(0)$ contains only finitely many isomorphism classes of analytic spaces.

The germs in Table 2a are right and contact simple by Proposition 10.1. The germs in Table 2b are not contact simple (and hence not right simple): $\tilde{E}_6$ is a family of cones over non-singular elliptic curves in $\mathbf{C}P^2$, $\tilde{E}_7$ is a family of four lines through the origin in $\mathbf{C}^2$, and $\tilde{E}_8$ is a family of three parabolas [Arnold 1; Siersma]. Note that the germs of Table 2c form one-dimensional families under right equivalence, but all members of the family are contact equivalent [Laufer 4; Siersma p. 54]. Clearly if a germ g right simplifies to f and f is not right simple, then g is not right simple; the same applies to contact equivalence.

Characterization B3. The germ f is right simple.

Characterization B4. The germ f is contact simple.

The equivalence of Characterizations B1 and B3 follows from Proposition 10.1 and the above remarks [Arnold 1]. Characterization B3 implies Characterization B4 by definition. Conversely, a contact simple germ f which is not right simple right simplifies to a germ in Table 2b (by Proposition 10.1), but these are not contact simple. Hence f must be right simple.

The classification of simple germs has recently been extended to complete intersections [Giusti]. The *modality* of a germ f is defined in [Arnold 3]. A right-simple germ is zero-modal; all right equivalence classes of 1 and 2-modal germs have been listed [Arnold 2, 3, 5]. Moduli of resolutions of two-dimensional normal singularities are studied in [Laufer 3, 4]. The following result provides a connection between Characterizations A2 and B3.

Theorem 11.1 [Randell]. *For almost all germs $f(x, y, z)$ (in the sense of the Newton diagram), the geometric genus p of $f^{-1}(0)$ is less than or equal to the modality of f.*