The equivalence of Characterizations C2 and C8 is proved in [Wagreich 2, p. 66; Saito 2, Theorem 1.9]. In fact, the zero loci of the germs $\tilde{E}_6$, $\tilde{E}_7$ and $\tilde{E}_8$ have minimal resolution as above with $E^2 = -3, -2$ and -1 respectively.

Characterization C9. The germ $f^{-1}(0)$ is isomorphic to $g^{-1}(0)$, where g is a weighted homogeneous polynomial with weights w_i satisfying $w_0^{-1} + ... + w_n^{-1} = n/2$.

The equivalence of Characterizations C2 and C9 is proved in [Saito 2, Satz 2.11]. In fact, the germs in Table 2b have the following weights:

GERM	WEIGHTS
P_8	(3, 3, 3)
X_9	(4, 4)
J_{10}	(3, 6)

Appendix II

The Monodromy Groups of the Minimal Hyperbolic Germs

PROPOSITION. *The monodromy groups of the germs P_9, X_{10}, and J_{11} have exponential growth.*

In this appendix, we present an (unpublished) proof of this proposition due to E. Looijenga. In fact, we will show that these groups have $PSL(2, \mathbf{Z})$ as subquotient (quotient of a subgroup). We let $0(V)$ denote the orthogonal group of a $\mathbf{Z}$- or $\mathbf{R}$-module V equipped with a bilinear form.

Suppose G is a polyhedral graph whose edges are weighed by non-zero integers. By convention, the weight 1 is omitted. Let L_G denote the free $\mathbf{Z}$-module generated by the vertices $v_1, ..., v_n$ of G. Define a symmetric bilinear form (,) on L_G by setting $(v_i, v_i) = -2$, and $(v_i, v_j) = 0$ if there is no edge from v_i to v_j, otherwise equal to the weight on this edge. Conversely, given a symmetric integral bilinear form (,) on a free module L with basis $\alpha_1, ..., \alpha_n$ with the property that $(\alpha_i, \alpha_i) = -2$ for all i, one associates a graph to it in the obvious way.

For $\alpha \in L_G$, let s_α (*reflection in* α) be the isometry of L_G defined by

$$s_\alpha(\beta) = \beta - 2\frac{(\alpha, \beta)}{(\alpha, \alpha)}\alpha$$

for $\beta \in L_G$. The *reflection group* $\mathcal{R}(G)$ of the graph G is defined to be the subgroup of $0(L_G)$ generated by $s_{\alpha_1}, \ldots, s_{\alpha_n}$.

Example 1. Consider a reduced irreducible root system in a vector space V. Let $\alpha_1, \ldots, \alpha_n$ be a collection of simple roots, let L be the free $\mathbf{Z}$-module spanned by the α_i, and let $(,)$ be the negative of an invariant bilinear form [Serre, Chapter 5]. If $(\alpha_i, \alpha_i) = -2$ for all i, then the corresponding graph must be of type A_k, D_k, E_6, E_7 or E_8. The reflection group of these graphs equals the Weyl group, the group generated by reflections in all the roots [Serre, p. V-16]. Furthermore, the reflection group together with the generators $s_{\alpha_1}, \ldots, s_{\alpha_n}$ forms a Coxeter system [Bourbaki, p. 92]. (A *Coxeter system* is a group G, a collection of elements $g_1, \ldots, g_n$ and a symmetric integral $n \times n$ matrix $\{m_{ij}\}$ with $m_{ii} = 1$ and $2 \leqslant m_{ij} \leqslant \infty$ for $i \neq j$, with the property that G is isomorphic to the free group with generators $g_1, \ldots, g_n$ and relations $(g_i, g_j)^{m_{ij}} = 1$, for all i, j.)

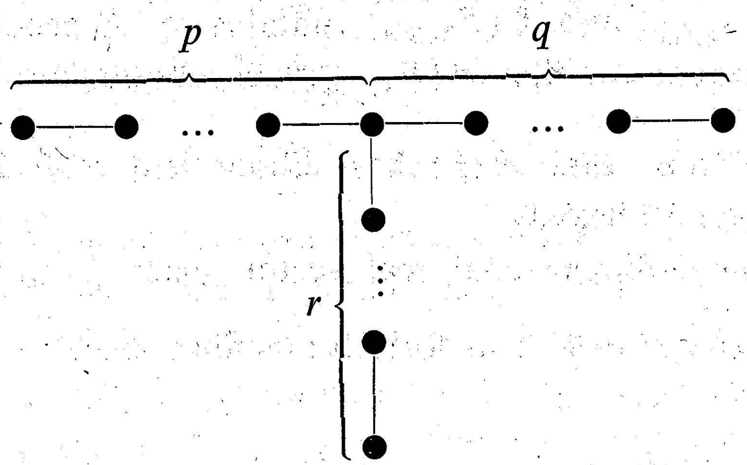
Example 2. The monodromy group of a germ f is the reflection group of a quadratic form diagram for f (Sections 13 and 14). When this diagram is a tree (which is only possible for the simple germs), its reflection group together with the generators $T_1, \ldots, T_\mu$ forms a Coxeter system. In general, this reflection group is not a Coxeter system [A'Campo 2, II, p. 403].

Lemma [Gabrielov 3]. If the graph G' is a subgraph of the graph G, then $\mathcal{R}(G')$ is a subquotient of $\mathcal{R}(G)$.

Proof. Let $\alpha_1', \ldots, \alpha_m'$ be a basis of $L_{G'}$ corresponding to the vertices of G', let $\alpha_1, \ldots, \alpha_m$ be the corresponding elements in L_G, and extend this to a basis $\alpha_1, \ldots, \alpha_m, \alpha_{m+1}, \ldots, \alpha_n$ of L_G corresponding to the vertices of G. The map $\alpha_i' \to \alpha_i$ is an isometric embedding of $L_{G'}$ in L_G. Let $\mathcal{R}'$ be the subgroup of $\mathcal{R}(G)$ generated by $s_{\alpha_1}, \ldots, s_{\alpha_m}$; it has a presentation with these generators and certain relations. Any relation among these s_{α_i} holds also for $s_{\alpha_i} \mid L_{G'} = s_{\alpha_i'}$. Thus $\mathcal{R}'$ maps onto $\mathcal{R}(G')$.

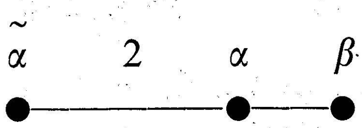
Fact. If a subquotient of a group G has exponential growth, then so does G.

Proof of Proposition. 1. A quadratic form diagram for the germs P_9, X_{10}, and J_{11} is given in column 5 of Table 2. These graphs contain a subgraph of the form $T_{3,3,4}$, $T_{2,4,5}$, and $T_{2,3,7}$ respectively, where $T_{p,q,r}$ is the graph

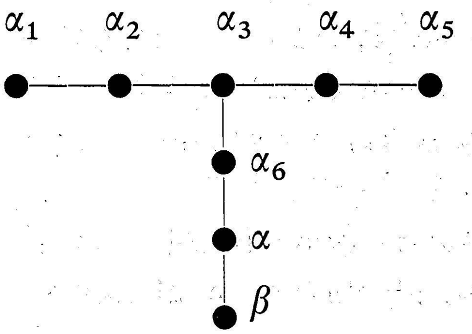


Hence it suffices to show that the reflection groups of these graphs have exponential growth.

Let Γ be the graph



with vertices corresponding to basis elements $\tilde{\alpha}$, α, β in L_Γ as indicated. We claim that $\mathscr{R}(T_{p,q,r})$ has $\mathscr{R}(\Gamma)$ as subquotient, for $(p, q, r) = (3, 3, 4)$, $(2, 4, 5)$, and $(2, 3, 7)$. Consider (for example) $T_{3,3,4}$, with vertices corresponding to basis elements $\alpha_i \in L_{T3,3,4}$ as indicated:



This contains the graph E_6. Let

$$\tilde{\alpha} = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 + \alpha_5 + 2\alpha_6 \in L_{T3,3,4}$$

be the largest root of E_6 [Bourbaki, p. 165]. Since all the roots of E_6 are the same length, $(\tilde{\alpha}, \tilde{\alpha}) = -2$. The lattice spanned by $\tilde{\alpha}$, α, and β has diagram Γ. The reflections s_α and s_β are in $\mathscr{R}(T_{3,3,4})$. We claim that $s_{\tilde{\alpha}}$ is in $\mathscr{R}(T_{3,3,4})$ as well: The restriction $s_{\tilde{\alpha}} \mid L_{E6}$ is in $\mathscr{R}(E_6)$, since E_6 is a root system. Hence $s_{\tilde{\alpha}} \mid L_{E6} = (s_{\alpha_{i(1)}} \circ \dots \circ s_{\alpha_{i(m)}}) \mid L_{E6}$ for some $1 \leqslant i(1), \dots, i(m) \leqslant 6$.

Also, $s_{\tilde{\alpha}}$ and $s_{\alpha i(1)} \circ \dots \circ s_{\alpha i(m)}$ are both the identity when restricted to the orthogonal complement of $L_{E_6} \otimes \mathbf{R}$ in $L_{T3,3,4} \otimes \mathbf{R}$. Thus $s_{\tilde{\alpha}} = s_{\alpha i(1)} \circ \dots \circ s_{\alpha i(m)}$, and $\mathscr{R}(T_{3,3,4})$ contains $s_{\tilde{\alpha}}$. A proof similar to that of the lemma then shows that $\mathscr{R}(T_{3,3,4})$ has subquotient $\mathscr{R}(\Gamma)$.

3. Next we show that $\mathscr{R}(\Gamma)$ has subquotient $PSL(2, \mathbf{Z})$. This uses [E. Artin, Chapter V] heavily.

Let V be the 3-dimensional real vector space $L_\Gamma \otimes \mathbf{R}$. The bilinear form $(\,,)$ of Γ extends to V. This form is indefinite since $\tilde{\alpha} + \alpha$ has length 0. Let

$$0'(V) = \{f \in 0(V)\colon \det f = 1 \text{ and spinor norm of } f \text{ equal to } 1\,\mathbf{R}^{*2}\}.$$

Since V is indefinite, it is known [E. Artin, p. 200] that

$$0'(V) \xrightarrow{\cong} PSL(2, \mathbf{R})\,. \tag{1}$$

Since $PSL(2, \mathbf{R})$ contains $PSL(2, \mathbf{Z})$ as a subgroup, the idea is to find elements of $\mathscr{R}(\Gamma) \subset 0(\Gamma)$ which are in $0'(V)$ and map to generators of $PSL(2, \mathbf{Z})$. The standard generators of $PSL(2, \mathbf{Z})$ are

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad T = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

with relations $S^2 = (ST)^3 = 1$. By inspection, it is found that the elements $s_{\tilde{\alpha}}\, s_\beta$ and $s_\beta\, s_\alpha$ of $0(L)$ satisfy $(s_{\tilde{\alpha}} s_\beta)^2 = (s_\beta s_\alpha)^3 = 1$, and have determinant equal to 1 and spinor norm equal to $1\mathbf{R}^{*2}$. Therefore we would like to choose the isomorphism (1) such that $s_{\tilde{\alpha}}\, s_\beta$ maps to S, and $s_{\tilde{\alpha}}\, s_\alpha = (s_{\tilde{\alpha}} s_\beta)^{-1} (s_\beta s_\alpha)$ maps to $S^{-1}(ST) = T$.

The isomorphism (1) is done in two steps. First, let $D_0(V)$ be the elements of the Clifford algebra of V of norm 1; then [E. Artin, p. 199]

$$D_0(V)/\{\pm 1\} \xrightarrow{\cong} 0'(V)\,. \tag{2}$$

We do not need to know exactly what this isomorphism is, but only that

$$v \circ w \to s_v s_w$$

for elements v, w in V regarded as a subspace of the Clifford algebra, and $v \circ w$ their product. Hence under the above isomorphisms

$$\frac{1}{2}\tilde{\alpha} \circ \beta \mapsto s_{\tilde{\alpha}} s_\beta\,, \qquad \frac{1}{2}\tilde{\alpha} \circ \alpha \mapsto s_{\tilde{\alpha}} s_\alpha\,. \tag{3}$$

Secondly [E. Artin, p. 199],

$$D_0(V)/\{\pm 1\} \cong PSL(2, \mathbf{R}). \tag{4}$$

We examine this more closely. Let

$$A_1 = \sqrt{2}(\alpha + \tilde{\alpha} + \beta/2), \quad A_2 = \tilde{\alpha}/\sqrt{2}, \quad A_3 = \beta/\sqrt{2}.$$

Then A_1, A_2, A_3 is an orthogonal basis of V, and the matrix of (,) with respect to this basis is the diagonal matrix $\langle +1, -1, -1 \rangle$. Let $C^+(V)$ be the subspace of the Clifford algebra of V spanned by the elements of even grading; $C^+(V)$ is generated by 1, i_1, i_2, i_3, where $i_1 = A_2 \circ A_3$, $i_2 = A_3 \circ A_1$, and $i_3 = A_1 \circ A_2$, and has multiplication table as in [E. Artin, top of p. 200] with $a = -1$. The map

$$C^+(V) \to M(2, \mathbf{R})$$

(where $M(2, \mathbf{R})$ is the algebra of all 2×2 matrices over $\mathbf{R}$) defined by

$$1 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad i_1 \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$i_2 \to \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad i_3 \to \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

is an isomorphism. (This is slightly different from the isomorphism of [E. Artin, p. 200].), and the restriction of this map to $D_0(V)$ gives the isomorphism (4). Furthermore,

$$\frac{1}{2}\tilde{\alpha} \circ \beta = i_1 \mapsto S, \quad \frac{1}{2}\tilde{\alpha} \circ \alpha = 1 - \frac{1}{2}(i_1 + i_3) \mapsto T \tag{5}$$

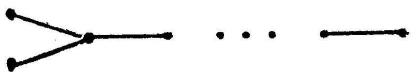
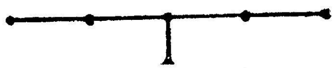
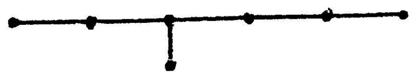
under this isomorphism. Combining isomorphisms (2) and (4) gives isomorphism (1), and (3) and (5) show that

$$s_{\tilde{\alpha}} s_\beta \mapsto S, \qquad s_{\tilde{\alpha}} s_\alpha \mapsto T$$

under isomorphism (1). Thus $\mathscr{R}(\Gamma)$ maps onto $PSL(2, \mathbf{Z})$, and hence has $PSL(2, \mathbf{Z})$ as subquotient.

4. Finally, $PSL(2, \mathbf{Z})$ is isomorphic to the free product $(\mathbf{Z}/2\mathbf{Z}) * (\mathbf{Z}/3\mathbf{Z})$ [Serre, ch. 7; Lehner, p. 59], which has exponential growth.

TABLE 1 RATIONAL DOUBLE POINTS

(1) $f(x, y, z)$	(2) (p, q, r)	(3) Dual Graph of Resolution	(4) Name	(5) Subgroup G of S^3	(6) Order of G	(7) Weights
$x^{k+1}+y^2+z^2$	$(1, 1, k)$	 k vertices	A_k $k \geqslant 1$	cyclic	$k+1$	$(k+1, 2, 2)$
$x^{k-1}+xy^2+z^2$	$(2, 2, k-2)$	 k vertices	D_k $k \geqslant 4$	binary dihedral	$4(k-2)$	$(k-1, 2(k-1)/(k-2), 2)$
$x^3+y^4+z^2$	$(2, 3, 3)$		E_6	binary tetrahedral	24	$(3, 4, 2)$
$x^3+xy^3+z^2$	$(2, 3, 4)$		E_7	binary octahedral	48	$(3, 9/2, 2)$
$x^3+y^5+z^2$	$(2, 3, 5)$		E_8	binary icosahedral	120	$(3, 5, 2)$

Column 1 lists the germ $f(x, y, z)$. Column 3 lists the dual graph of the minimal resolution of $f^{-1}(0)$. The name of the graph is given in column 4. Each graph is of type $T_{p,q,r}$, where (p, q, r) are listed in column 2. Every vertex of the graph represents a nonsingular rational curve of self-intersection -2. The analytic set $f^{-1}(0)$ is isomorphic to $\mathbf{C}^2/G$, where G is the finite subgroup of S^3 listed in column 5. Each germ f is weighted homogeneous, with weights as listed in column 7.

TABLE 2 GERMS OF LOW MILNOR NUMBER

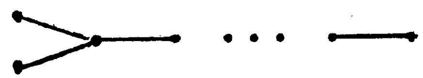
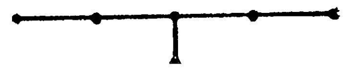
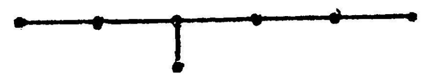
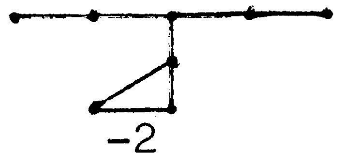
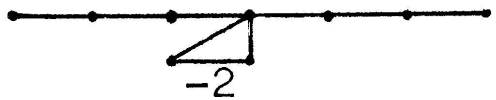
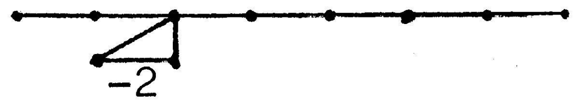
(1)	(2) Name	(3) μ	(4) $f(z_0, \dots, z_n)$	(5) Quadratic form diagram
a	$A_k,\ k \geqslant 1$	k	z_0^{k+1}	 k vertices
(simple germs)	$D_k,\ k \geqslant 4$	k	$z_0^{k+1} + z_0 z_1^2$	
	E_6	6	$z_0^3 + z_1^4$	
	E_7	7	$z_0^3 + z_0 z_1^3$	
	E_8	8	$z_0^3 + z_1^5$	
b	P_8 or $\tilde{E}_6$	8	$z_0^3 + z_1^2 z_2 + a z_0 z_2^2 + b z_2^3,\ 4a^3 + 27b^2 \neq 0$	
(almost-simple germs)	X_9 or $\tilde{E}_7$	9	$z_0 z_1 (z_0 - z_1)(z_0 - a z_1),\ a \neq 0, 1$	
	J_{10} or $\tilde{E}_8$	10	$z_0 (z_0 - z_1^2)(z_0 - a z_1^2),\ a \neq 0, 1$	

TABLE 2 *(continuation)*

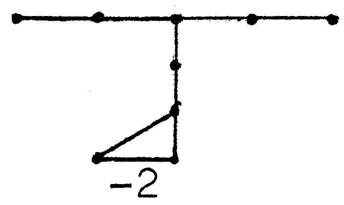
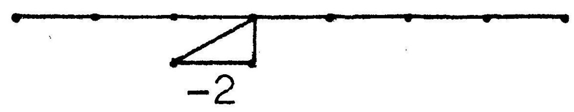
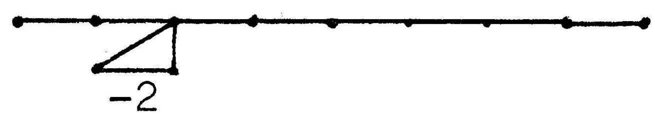
c	P_9 or $T_{3,3,4}$	9	$z_0^2z_2+z_1^3+z_1^2z_2+az_2^4$, $a \neq 0$	
(minimal hyperbolic germs)	X_{10} or $T_{2,4,5}$	10	$z_0^4+z_0^2z_1^2+az_1^5$, $a \neq 0$	
	J_{11} or $T_{2,3,7}$	11	$z_0^3+{}_\varepsilon z_0^2z_1^2+az_1^7$, $a \neq 0$, $_\varepsilon = \pm 1$	

Table 2 is divided into parts a, b, and c as indicated in column 1. These parts are referred to in the text as Table 2a, etc. Column 4 contains the equation of the germ; to each equation must be added the quadratic form $z_r^2 + \ldots + z_n^2$ in the variables $z_r, \ldots, z_n$ not occurring in the equation. Column 2 gives the type of the germ in Arnold's notation, column 3 gives its Milnor number μ, and column 5 gives a quadratic form diagram.

TABLE 3

SIMPLIFICATION TABLE

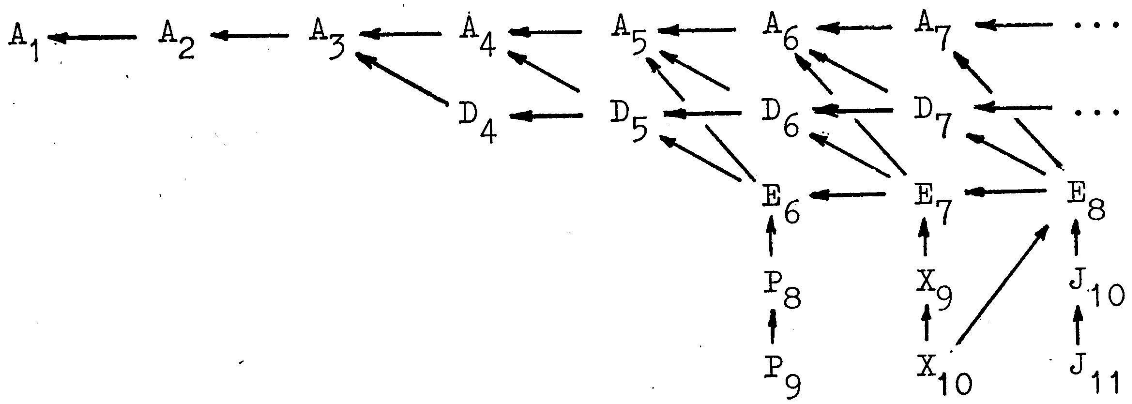


Table 3 lists some (but not all) of the simplifications that occur among the germs of Table 2.

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