TERMWISE AVERAGES OF TWO DIVERGENT SERIES

TERMWISE AVERAGES OF TWO DIVERGENT SERIES

by J. Marshall Ash and Harlan Sexton

Definition. For $a, b > 0$, let $M_\infty(a, b) = \max\{a, b\}$, $M_r(a, b) = \left(\frac{a^r + b^r}{2}\right)^{\frac{1}{r}}$ for finite non-zero r, $M_0(a, b) = \sqrt{ab}$, and $M_{-\infty}(a, b) = \min\{a, b\}$.

Definition. The sequence $\{a_n\}$, $n = 1, 2, \ldots$, is *convex* if $a_{n+2} - 2a_{n+1} + a_n \geqslant 0$, n $= 1, 2, \ldots$.

If $\{a_n\}$ is convex then the union of the line segments $\overline{(n, a_n)\,(n+1, a_{n+1})}$, $n = 1, 2, \ldots$, is the graph of a continuous convex function on the interval $[1, \infty)$.

Let $\{a_n\}$, $\{b_n\}$ be sequences of positive numbers. If $\sum a_n$ and $\sum b_n$ are finite, so is $\sum_n M_r(a_n, b_n)$ since $\sum M_\infty(a_n, b_n) < \sum(a_n + b_n) < \infty$ and $M_r(a, b)$ is an increasing function of r, $-\infty \leqslant r \leqslant \infty$ [Hardy, Littlewood, Polya, *Inequalities*, Cambridge Univ. Press, Cambridge (1973), pp. 15, 26]. If either $\sum a_n = \infty$ or $\sum b_n = \infty$ and if $r > 0$, then $\sum_n M_r(a_n, b_n) = \infty$ since

$$\sum M_r(a_n, b_n) = \sum\left(\frac{a_n^r + b_n^r}{2}\right)^{\frac{1}{r}} \geqslant 2^{-1/r} \sum M_\infty(a_n, b_n)$$

$$\geqslant 2^{-1/r} \max\{\Sigma a_n, \Sigma b_n\} = \infty .$$

If however, $r \leqslant 0$, the situation is entirely different.

THEOREM. *There are convex monotonically decreasing to zero sequences* $\{a_n\}$, $\{b_n\}$ *with* $\sum a_n = \sum b_n = \infty$, *such that* $\sum_n M_r(a_n, b_n) < \infty$ *for* $-\infty \leqslant r \leqslant 0$.

Remark 1. The most interesting special cases are $r = -\infty$ where the conclusion is $\sum \min\{a_n, b_n\} < \infty$ and $r = 0$ where the conclusion is $\sum \sqrt{a_n b_n} < \infty$. Since $\lim_{r \to 0+} M_r(a, b) = M_0(a, b)$ [ibid, p. 15] the theorem coupled with its preceding remarks form a classification with a tidy "dividing line".

Remark 2. It is only the monotonicity and the convexity that make the theorem interesting, for $a_{2n-1} = b_{2n} = n^{-1}$, $a_{2n} = b_{2n-1} = 2^{-n}$, $n = 1, 2, \ldots$ implies $\sum a_n = \sum b_n = \infty$ and $\sum \sqrt{a_n b_n} < \infty$.

Remark 3. If $a_n \nearrow \infty$ and $b_n \nearrow \infty$ and $\sum a_n^{-\delta} = \sum b_n^{-\delta} = \infty$ for some $\delta > 0$, then $\sum (a_n + b_n)^{-\delta}$ may be finite. This follows from the theorem and the observation that $(a_n + b_n)^{-\delta} \leqslant M_{-\infty} (a_n^{-\delta}, b_n^{-\delta})$. The attempt to prove this (with $\delta = \frac{1}{2}$) was the motivation for this note.

Remark 4. The theorem (with $r = 0$) shows that Cauchy's inequality—$(\sum a_n b_n)^2 \leqslant \sum a_n^2 \sum b_n^2$—has no converse in the sense that the finitude of the smaller side does not imply the finitude of either term on the larger side, even for fairly regular sequences. Hölder's inequality may be treated similarly.

Proof. Since $M_r(a, b) \nearrow$ as $r \nearrow 0$ we need only produce convex monotonically decreasing to zero divergent series $\sum a_n$, $\sum b_n$ with the property that $\sum \sqrt{a_n b_n} < \infty$.

We begin by constructing two divergent sequences $\{\alpha_n\}$, $\{\beta_n\}$, each of which will have blocks of constancy, be nonincreasing to zero, and have a graph which lies above and contains the corners of its convex hull. Furthermore $\sum \sqrt{\alpha_n \beta_n}$ will be finite. Pictorially, $\{\alpha_n\}$ will look like this:

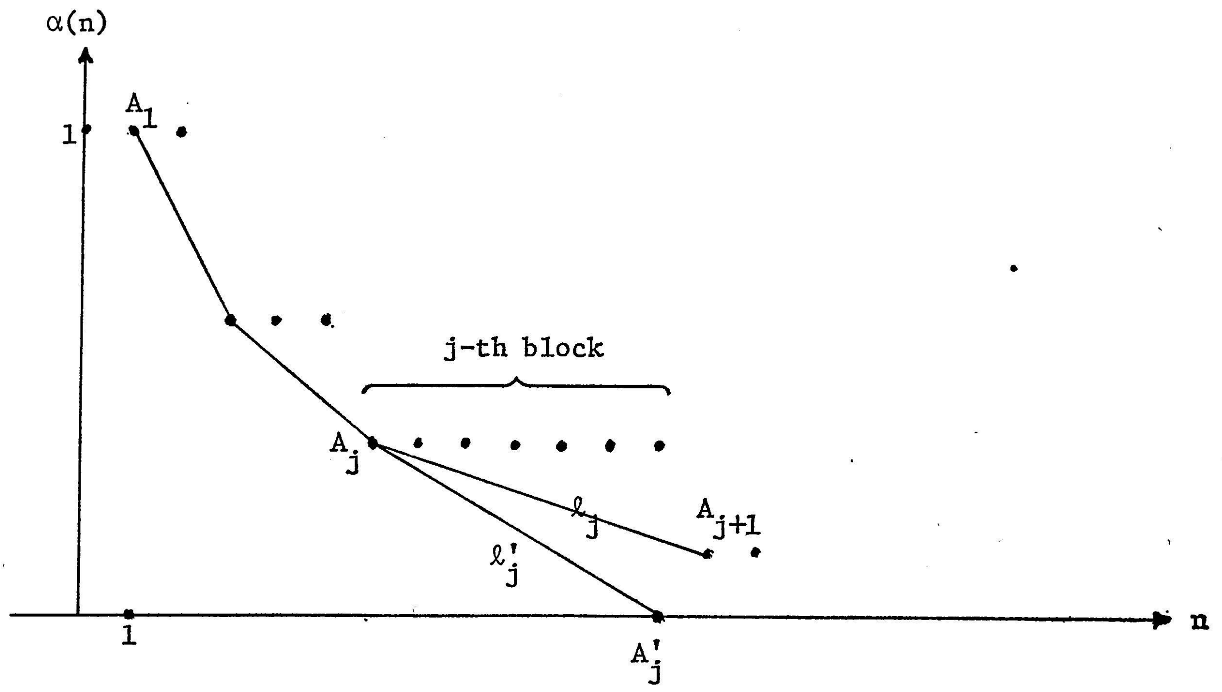

Figure 1

and similarly B_j will be the first point of the j-th block of β's, B_j' will lie on the axis below the last point of the β's j-th block, m_j will be the line connecting B_j with B_{j+1}, and m_j' will be the line connecting B_j with B_j'.

We will construct the α's and β's one block at a time. Let $\{c_n\}$ be any convergent sequence with all $c_n > 0$. Let $\alpha_1 = \alpha_2 = 1$ and $\beta_1 = c_1^2$ so that the first block of α's has length 2 and the first block of β's has length 1. Continue inductively as follows. We suppose that after the n-th stage n blocks of α's and β's have been chosen so that

(1) $\sum \alpha \geqslant n + 1$, $\sum \beta \geqslant n - 1$, $\alpha \searrow$, $\beta \searrow$

(2) $A_n' > B_n' > A_{n-1}'$ (We identify the point A_n' with its first coordinate; here A_0' is taken to be zero.)

(3) $\displaystyle\sum_{j=1}^{B_n'} \sqrt{\alpha_j \beta_j} \leqslant c_1 + \ldots + c_n$

(4) The polygonal paths $l_1\, l_2 \ldots l_{n-1}\, l_n'$ and $m_1\, m_2 \ldots m_n'$ are convex.

To reach the next stage of the construction first pick $A_n' - B_n'$ β's all equal to B where $B > 0$ is so small that $\displaystyle\sum_{j=B_n'+1}^{A_n'} \sqrt{\alpha_j B} \leqslant \frac{1}{2} c_{n+1}$ and so small that $B < \beta_{B_n'}$. Then pick sufficiently many more β's of this same size B so that there are now more β's than α's, and so that $\sum \beta \geqslant n$, and so that the path $m_1 \ldots m_n\, m_{n+1}'$ is convex. In much the same manner we now pick $B_{n+1}' - A_n'$ α's all equal to A where A is so small that $\displaystyle\sum_{j=A_n'+1}^{B_{n+1}'} \sqrt{AB} = (B_{n+1}' - A_n') \sqrt{AB} \leqslant \frac{1}{2} c_{n+1}$ and so small that $A < \alpha_{A_n'}$. Then pick sufficiently many more α's of this same size A so that there are now more α's than β's, and so that $\sum \alpha \geqslant n + 2$, and so that the path $l_1 \ldots l_n\, l_{n+1}'$ is convex. Now (1)-(4) hold with n replaced by $n + 1$.

Following a suggestion of Andrejs Treibergs we complete the proof as follows. Define the sequence $\{a_k\}$ $(\{b_k\})$ as the projection of the α's (β's) down onto their supporting lines l_j (m_j).

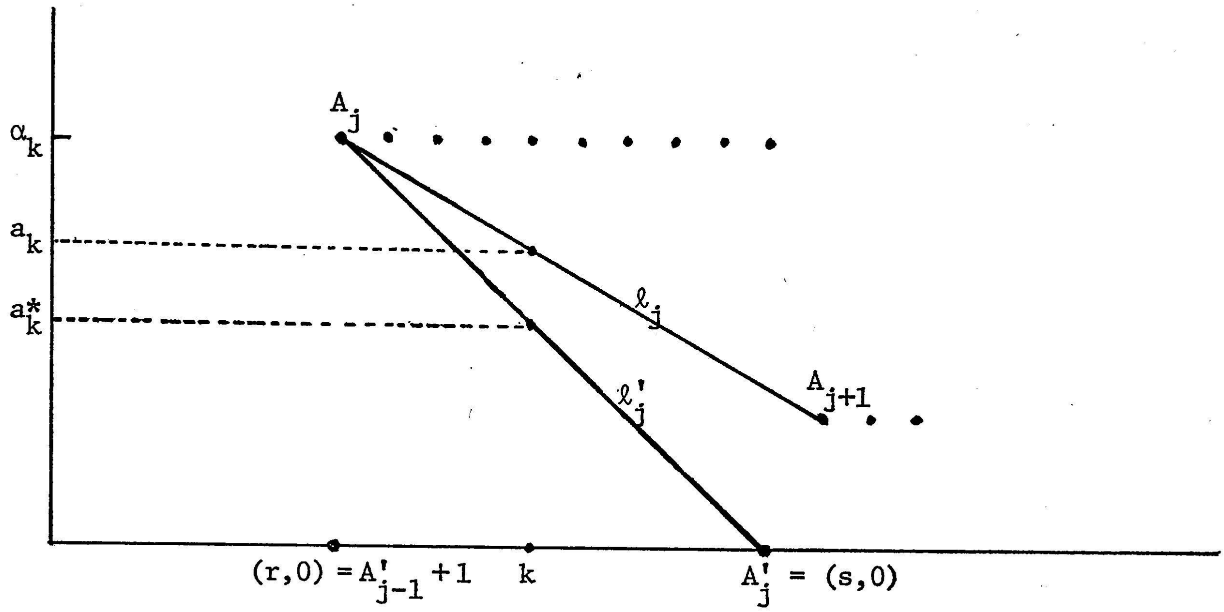

Figure 2

Then $\sum \sqrt{a_k\, b_k} < \sum \sqrt{\alpha_k\, \beta_k} \leqslant \sum c_n < \infty$. Clearly $\{a_k\}$ and $\{b_k\}$ are convex and monotonically decreasing to zero, and $\sum a_k > \sum a_k^* = \frac{1}{2} \sum \alpha_k = \infty$ since $\sum_{k=r}^{s} a_k^* = \frac{1}{2} \sum_{k=r}^{s} \alpha_k$. Similarly, $\sum b_k = \infty$.

(Reçu le 29 juin 197

J. Marshall Ash
Harlan Sexton

Department of Mathematics
De Paul University
2323 North Seminary Av.
Chicago, Illinois 60614

Stanford University
Stanford, California 94305