§4. Proof of the lemmas

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Otherwise

$$\gamma_{\mathfrak{p}}^{\chi}(2q) = 1\,. \tag{15}$$

Remark. The condition on $\det \tilde{q}$ implies non-degeneracy of q at $\mathfrak{p}$.

§ 3. Proof of the main theorem

Note that the rank of q is even because determinant of the associated bilinear form is odd. Therefore

$$\gamma_{v}^{\chi}(aq) = \left(\frac{a}{(\det \tilde{q})}\right) \gamma_{v}^{\chi}(q) \tag{16}$$

for *any* character χ.

Now let us apply the Weil reciprocity law for the character χ with support in dyadic components equal to the integers in the corresponding ring, and to the forms q and $2q$.

We have

$$\prod_{v} \gamma_{v}^{\chi}(2q) = 1$$

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For an archimedian components we have $\gamma_{v}^{\chi}(2q) = \gamma_{v}^{\chi}(q)$ because both depend only on the signatures. Therefore dividing those two identities, and using lemma 2 and (16) we obtain the identity (4).

Remark. Levine's lemma which in a specialization of the theorem for $R = \mathbf{Z}$ in fact follows from Milgram's formula (12). We should not worry about ramification. Therefore lemma 1 can be used for the character χ_0 and is actually a classical property of Gauss sums ([2]). Lemma 2 in this case essentially contains in [1].

§ 4. Proof of the lemmas

Proof of lemma 1. The Witt group of quadratic forms over a field of zero characteristic is generated by one-dimensional forms ([4]). Because γ^{χ} is a character of the Witt group it is enough to check the lemma for forms of one variable. Let π be a local parameter. Suppose that $q(x) = \alpha\, \pi^{b}\, x^{2}$, and $\chi(x) = \exp(2\pi i \operatorname{Tr}(\beta\pi^c x))$ where α and β are units. Suppose that the different $\mathfrak{d}$ of $F_{\mathfrak{p}}/\mathbf{Q}_p$ is $\mathfrak{d} = (\pi^d)$. Let n be an integer such that $2n + b + c + d \geqslant 0$. Then we can take as L the lattice (π^n). The dual lattice $L^{\#}$ is $(\pi^{-b-c-n+d})$. Therefore

$$\gamma^{\chi}(q) = \sum_{x \in (\pi^{-b-c-n-d})/(\pi^n)} \exp 2\pi i (\operatorname{Tr} \alpha\beta\pi^{c+b}x^2) \Big/ \Big| \sum_{x \in (\pi^{-b-c-n-d})/(\pi^n)} \exp 2\pi i (\operatorname{Tr} \alpha\beta\, \pi^{b+c}x^2) \Big|$$

After a change of variables $\pi^{-b-c-n-d}\, y = x$, we obtain

$$(17) \qquad \gamma^{\chi}(q) = \left(\sum_{y \in R_{\mathfrak{p}}/(\pi^{2n+b+c+d})} \exp 2\pi i \operatorname{Tr}\left(\frac{\alpha\beta y^2}{\pi^{2n+b+c+d}\pi^d}\right)\right) \Big/ \left| \sum_{y \in R_{\mathfrak{p}}/(\pi^{2n+b+c+d})} \exp 2\pi i \operatorname{Tr}\left(\frac{\alpha\beta y^2}{\pi^{2n+b+c+d}\pi^d}\right)\right|$$

The numerator of $\gamma^{\chi}(q)$ is a Gauss sum of the type considered by Hecke [2]. The same arguments show that

$$(18) \qquad \gamma^{\chi}(aq) = \left(\frac{a}{\pi^{2n+b+c+d}}\right) \gamma^{\chi}(q)$$

Now the support of χ is (π^{c+d}), $(\det \tilde{q})\, R_{\mathfrak{p}} = (\pi^b)$, and $\left(\frac{a}{\pi^{2n}}\right) = 1$, therefore the lemma follows.

Now let F_{2^f} denote a field of 2^f elements. Let $\tilde{\chi}$ denote a non-trivial character of the additive group of F_{2^f}. There exist a canonical choice of $\tilde{\chi}$, namely

$$(19) \qquad \tilde{\chi}_0(x) = (-1)^{\operatorname{Tr}_{F_{2^f} | F_2}(x)}$$

Note that kernal of $\tilde{\chi}_0$ is an additive subgroup of elements of the form $x + x^2$.

LEMMA 3. Let $\bar{q}$ be a non singular quadratic form defined on a vector space V over F_{2^f}. Then

$$(20) \qquad \gamma^{\tilde{\chi}}(\bar{q}) = \sum_{x \in V} \tilde{\chi}(\bar{q}(x)) \Big/ \Big| \sum_{x \in V} \tilde{\chi}(q(x)) \Big|$$

is equal to the Arf invariant of $\bar{q}$.

Proof. Both Gauss sum (20) and the Arf invariant are characters of the Witt group of quadratic forms. Therefore it is enough to check the statement for binary quadratic forms. But the number of elements in $F_{2^f} \oplus F_{2^f}$ on which $\tilde{\chi}\,(\bar{q}\,(x))$ takes the value 1, is $2^{2f-1} + (\operatorname{Arf} \bar{q})\, 2^{f-1}$ and the number of elements where $\tilde{\chi}\,(\bar{q}\,(x))$ takes the value -1 is $2^{2f-1} - (\operatorname{Arf} \bar{q})\, 2^{f-1}$. Indeed it is easy to check for form of Arf invariant 1 (which is $q\,(\alpha, \beta) = \alpha\beta$). On the other hand form of Arf invariant -1 can be written as $q\,(\alpha, \beta) = \alpha^2 + \alpha\beta + s\beta^2$, where $s \neq \gamma + \gamma^2$ for any γ. ([4]). But if m is chosen in such a way that $\tilde{\chi}\,(x) = \tilde{\chi}_0\,(mx)$ ($\tilde{\chi}_0$ is defined above), we have

$$\tilde{\chi}\,(\alpha^2 + \alpha\beta + s\beta^2) = -\,\tilde{\chi}_0\,((s + m^2\alpha^2)\,(1 + m^2\beta^2))$$

i.e. number of the elements for which $\tilde{\chi}\,(\bar{q}\,(x)) = -1$ in a quadratic space with Arf invariant -1, equals the number of elements for which $\tilde{\chi}_0\,(q\,(x)) = 1$ in the space with Arf invariant 1. Therefore the lemma 3 follows.

Remark. A connection between the Arf invariant and Gauss sums was first observed in [1].

COROLLARY. *Let $F_\mathfrak{p}$ be a dyadic local field. Let χ denote a character of the additive group of $F_\mathfrak{p}$ with support $R_\mathfrak{p}$. Let q be an integer quadratic form on the $R_\mathfrak{p}$-module V such that the determinant of the associated bilinear form is a unit. Then*

$$\sum_{x \in V/\pi V} \chi\left(q\left(\frac{x}{\pi}\right)\right) \Big/ \left|\, \sum \chi\left(q\left(\frac{x}{\pi}\right)\right)\right| \tag{21}$$

is equal to the Arf *invariant of* q mod $\mathfrak{p}$.

Proof. The map $x \to \chi\left(\frac{x}{\pi}\right)$ defines a non-trivial character of $R_\mathfrak{p}/\pi R_\mathfrak{p}$. Therefore the expressions (20) and (21) coincide.

LEMMA 4. Let χ denotes a character of the dyadic field $F_\mathfrak{p}$ with support $R_\mathfrak{p}$.

Let q denote an integer quadratic form over the $R_\mathfrak{p}$ module V such that determinant of the associated bilinear form is a unit. Then

$$\sum_{x \in V/\pi^n V} \chi\left(\frac{q\,(x)}{\pi^n}\right) = N^{\dim V} \sum_{x \in V/\pi^{n-2} V} \left(\frac{q\,(x)}{\pi^{n-2}}\right) \tag{22}$$

where N is the norm of the prime ideal of $R_\mathfrak{p}$.

Proof. We follow the classical scheme in [2]. Let $x = x_1 + \pi^{n-1} x_2$, where $x_2 \in V/\pi V$ and $x_1 \in V/\pi^{n-1}$. Then

$$\sum_{x \in V/\pi^n V} \chi\left(\frac{q(x)}{\pi^n}\right) = \sum_{\substack{x_1 \in V/\pi^{n-1} V \\ x_2 \in V/\pi V}} \chi\left(\frac{q(x_1) + \pi^{n-1}\tilde{q}(x_1, x_2) + \pi^{2n-2} q(x_2)}{\pi^n}\right)$$

$$= N \sum_{\substack{x_1 \equiv 0(\pi) \\ x_1 \in V|\pi^{n-1} V}} \chi\left(\frac{q(x_1)}{\pi^n}\right) + \sum_{x_1} \chi\left(\frac{q(x_1)}{\pi}\right)\left(\sum_{\substack{x_1 \not\equiv 0(\pi) \\ x_1 \in V/\pi^{n-1} V \\ x_2 \in V/\pi V}} \chi\left(\frac{\tilde{q}(x_1, x_2)}{\pi}\right)\right)$$

The sum in brackets is the sum of the values of the non-trivial (because det $\tilde{q}$ is unit and supp $\chi = R_{\mathfrak{p}}$) character, hence is equal zero. Therefore we obtain the result of the lemma because

$$\sum_{\substack{x_1 \equiv 0(\pi) \\ x_1 \in V/\pi^{n-1} V}} \chi\left(\frac{q(x)}{\pi^n}\right) = \sum_{x \in V/\pi^{n-1} V} \chi\left(\frac{q(x)}{\pi^{n-2}}\right)$$

Now we are ready to conclude the

Proof of lemma 2. Let e denote the ramification index of $F_{\mathfrak{p}}$ over $\mathbf{Q}_2$. Thus for a character with the support $R_{\mathfrak{p}}$ the dual of integer lattice V with respect to form $2q$ in the lattice $\frac{1}{\pi^e} V$. Hence

$$\gamma^\chi(2q) = \sum_{x \in \frac{1}{\pi^e} V/V} \chi(2q(x)) \Big/ \Big| \sum_{x \in \frac{1}{\pi^e} V/V} \chi(2q(x)) \Big| \tag{23}$$

$$= \sum_{x \in V/\pi^e V} \chi\left(\frac{q(x)}{\pi^e}\right) \Big/ \Big| \sum_{x \in V/\pi^e V} \chi\left(\frac{q(x)}{\pi^e}\right) \Big|$$

If e is odd then by lemma 4, (23) is equal to

$$\sum_{x \in V/\pi V} \chi\left(\frac{q(x)}{\pi}\right) \Big/ \Big| \sum_{x \in V/\pi V} \chi\left(\frac{q(x)}{\pi}\right) \Big|$$

which is, by the corollary to lemma 3, the Arf invariant of q mod $\mathfrak{p}$. If e is even then it follows from (22) that (23) is equal 1. This concludes the proof of lemma 2.