5. Norm conditions and commutativity

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conditions can be considered for algebras with or without identity. As we shall see later B. Aupetit [9] has obtained some theorems of this kind.

By considering the incomplete normed algebra of polynomials with complex coefficients and norm $\left\| \sum_{i=0}^{n} c_i x^i \right\| = \sum_{i=0}^{n} | c_i |$, it is easy to see that completeness is essential in Edwards' theorem.

5. Norm conditions and commutativity

The investigation of further conditions on the norm of a normed algebra lead to consideration of the *spectral radius*, defined by

$$\rho(x) = \lim_{n \to \infty} \| x^n \|^{1/n} .$$

We list several of its properties for later reference (see [73, pp. 10, 30]). For elements x and y in a normed algebra A:

(1) $\rho(x) \leqslant \| x \|$.

(2) $\rho(xy) = \rho(yx)$ and $\rho(x^n) = \rho(x)^n$.

(3) $\rho(x+y) \leqslant \rho(x) + \rho(y)$ and $\rho(xy) \leqslant \rho(x)\rho(y)$ if $xy = yx$.

(4) $\rho(x) = \sup \{ | \lambda | : \lambda \in \sigma(x) \}$ provided A is complete.

In view of the exceptionally strong consequences of the multiplicative norm condition $\| xy \| = \| x \| \, \| y \|$ it is natural to inquire into the algebraic implications of the similar condition

$$\| x^2 \| = \| x \|^2 . \qquad (*)$$

One familiar consequence of (*) in a normed algebra A is that

$$\rho(x) = \| x \| \qquad (**)$$

for all x in A. On the other hand since ρ always satisfies $\rho(x^2) = \rho(x)^2$, the conditions (*) and (**) are equivalent in any normed algebra. Their importance can be surmised by noting that they imply the Gelfand representation for commutative Banach algebras is isometric.

Almost simultaneously Claude Le Page [54] and R. A. Hirschfeld together with W. Żelazko [47] discovered independently that (*) in fact implies the commutativity of A. Although both papers use essentially the same techniques, the exponential function and Liouville's theorem, they are complementary in the direction of corollaries pursued. We shall survey these two papers here providing some simplifications of proofs and generalizations of the theorems when possible.

THEOREM 5.1. [Le Page, Hirschfeld-Żelazko]. *If A is a normed algebra over* $\mathbf{C}$ *satisfying* $\| x \|^2 \leqslant \alpha \| x^2 \|$ *for all* x *in* A *and some constant* α, *then* A *is commutative.*

Proof. The inequality $\| x \|^2 \leqslant \alpha \| x^2 \|$ extends to the completion of A so that we may assume A is a Banach algebra. Iterating the given estimate, $\| x \|^{2n} \leqslant \alpha^{2n-1} \| x^{2n} \|$ or $\| x \| \leqslant \alpha^{1-1/2n} \| x^{2n} \|^{1/2n} \to \alpha\rho(x)$ as $n \to \infty$.

If A has no identity, adjoin one to obtain A_1, which contains A isometrically as a closed two-sided ideal. (We do not claim that $\| x \| \leqslant \alpha\rho(x)$ holds on all of A_1.) Let x and y be arbitrary elements of A, and set $z(\lambda) = \exp(\lambda x)\, y \exp(-\lambda x)$, $\lambda \in C$. If L is any continuous linear functional on A, define $f(\lambda) = L(z(\lambda))$. Then f is an entire function of λ. Since $\exp(-\lambda x) = \exp(\lambda x)^{-1}$ (in A_1 if A has no identity element) and $z(\lambda) \in A$, $\| z(\lambda) \| \leqslant \alpha\rho(\exp(\lambda x)\, y \exp(-\lambda x)(= \alpha\rho(y)$. Therefore f is bounded and entire. By Liouville's Theorem f is constant so $f(\lambda) = L(y) = L(z(0))$. Differentiating f with respect to λ by the product rule and setting $\lambda = 0$, we have $0 = f'(0) = L(\exp(\lambda x)(xy - yx)\exp(-\lambda x))\,|_{\lambda=0} = L(xy - yx)$. By the Hahn-Banach Theorem, $xy - yx = 0$ since L was arbitrary.

As a corollary we obtain a relation between the spectral radius and the norm which implies commutativity. This is the form which the principal theorem of Hirschfeld-Żelazko takes.

COROLLARY 5.2. [Hirschfeld-Żelazko]. *If A is a complex normed algebra satisfying* $\| x \| \leqslant \alpha\rho(x)$ *for all* x *in* A *and some constant* α, *then* A *is commutative.*

Proof. Squaring the relation $\| x \| \leqslant \alpha\rho(x)$, we have $\| x \|^2 \leqslant \alpha^2\rho(x)^2 = \alpha^2\rho(x^2) \leqslant \alpha^2 \| x^2 \|$; so the hypothesis of Theorem 5.1 holds with the constant α^2.

Corollaries 5.3-5.7 below comprise the remaining results of the Hirschfeld-Żelazko paper and follow fairly easily from Corollary 5.2. An element of a normed algebra is called *quasi-nilpotent* if its spectral radius is zero.

COROLLARY 5.3. *If A is a complex normed algebra in which 0 is the only quasi-nilpotent element and ρ is subadditive and submultiplicative, then A is commutative.*

Proof. Under the hypotheses ρ is a norm on A. Apply Theorem 5.1 to the normed algebra (A, ρ) with $\alpha = 1$.

In Corollary 5.3 if A is complete we can omit the hypothesis that ρ is submultiplicative, and follow the *proof* of Theorem 5.1: Take L to be any ρ-continuous (hence $\|\cdot\|$-continuous) linear functional on A and observe that $z(\lambda)$ is ρ-bounded. Applying the Hahn-Banach Theorem to the normed linear space (A, ρ) we obtain $xy = yx$ as before.

COROLLARY 5.4. *If A is a complex Banach algebra and ρ is subadditive and submultiplicative, then $\rho(xy - yx) = 0$ for all x, y in A.*

Proof. Since $\rho(xy) \leqslant \rho(x)\rho(y)$, $N = \{x \in A : \rho(x) = 0\}$ is an ideal of A. N is closed because ρ is continuous, being subadditive and dominated by $\|\cdot\|$. Thus A/N is a Banach algebra. The spectral radius on A/N satisfies $\rho(x + N) = \rho(x)$ for every x in A since $\rho(x) = \sup\{|\lambda| : \lambda \in \sigma(x)\}$ and each element of N is quasi-regular [See Lemma 6.1 below.] Thus ρ on A/N is likewise subadditive and submultiplicative. Corollary 5.3 implies that A/N is commutative so that $xy - yx \in N$ as required.

In order to state the next corollary we need to define the (Jacobson) radical of an algebra, which will also play a prominent role in the latter part of this paper. A left ideal I of an algebra A is called *modular* if there is an element u in A satisfying $xu - x \in I$ for all x in A. The *radical* of A, denoted $\mathrm{Rad}(A)$ is the intersection of all maximal modular left ideals of A. Some relevant facts about the radical at this point are that $\mathrm{Rad}(A) \subset N = \{x \in A : \rho(x) = 0\}$ and $\mathrm{Rad}(A)$ is the largest two-sided ideal of A in which every element is quasi-regular (See Rickart [73, pp. 55-57].)

COROLLARY 5.5. *If A is a complex Banach algebra and ρ is subadditive and submultiplicative, then $A/\mathrm{Rad}(A)$ is commutative.*

Proof. Since N is an ideal under the given conditions, the properties of the radical mentioned above imply that $N = \mathrm{Rad}\, A$. An application of Corollary 5.4 yields the desired result.

We shall see subsequently that either the subadditivity or the submultiplicativity of ρ separately imply that $A/\mathrm{Rad}(A)$ is commutative.

COROLLARY 5.6. *In a non-commutative complex normed algebra,* $\inf\limits_{x \neq 0} \rho(x/\|x\|) = 0$ *and* $\inf\limits_{x \neq 0} \|x^2\|/\|x\|^2 = 0$.

Proof. If either infinum is positive, an inequality of the kind $\| x \| \leqslant \alpha \rho (x)$ or $\| x \|^2 \leqslant \alpha \| x^2 \|$ would hold, implying commutativity.

COROLLARY 5.7. *Every non-commutative finite dimensional normed algebra contains a non-zero nilpotent element.*

Proof. Since the unit sphere is compact the continuous function $x \to \| x^2 \|$ assumes the value of its infimum.

Le Page's study of commutativity considers conditions on the norm directly rather than on the spectral radius.

THEOREM 5.8. [Le Page]. *If A is a complex normed algebra with identity such that $\| xy \| \leqslant \alpha \| yx \|$ for all x, y in A and some constant α, then A is commutative.*

Proof. The norm condition extends to the completion of A so we may assume that A is a Banach algebra. Let $z(\lambda) = \exp(\lambda x)\, y \exp(-\lambda x)$ for λ in $\mathbf{C}$. Then $\| z(\lambda) \| \leqslant \alpha \| y \|$ so the proof is completed as in Theorem 5.1.

This theorem has been improved by Baker and Pym [27] to require that A have only a bounded approximate identity. Their method uses the exponential function and Liouville's Theorem in much the same way as Le Page. We do not know if 5.8 holds without an identity assumption.

THEOREM 5.9. [Le Page]. *If A is a normed algebra with identity and $a \in A$ satisfies $\| (a+\lambda) x \| \leqslant \| x (a+\lambda) \|$ for every x in A and λ in $\mathbf{C}$, then a lies in the center of A.*

Proof. The inequality extends to the completion so we may again assume A is complete. For any λ satisfying $| \lambda | > \| a \|$, put $x = y(a+\lambda)^{-1}$. Then $(a+\lambda) x = (a+\lambda)\, y\, (a+\lambda)^{-1} = (a/\lambda+1)\, y\, (a/\lambda+1)^{-1}$ so $\| (1+a/\lambda)\, y\, (1+a/\lambda)^{-1} \| \leqslant \| y \|$. Thus if μ is any complex number and n is an integer such that $n > \| a \| \cdot | \mu |$, then $\| (1+\mu/n)\, y\, (1+\mu/n)^{-1} \| \leqslant \| y \|$. Iterating this estimate n times, $\| (1+\mu/n)^n\, y\, (1+\mu/n)^{-n} \| \leqslant \| y \|$, so when $n \to \infty$ we have $\| \exp(\mu a)\, y \exp(-\mu a) \| y \|$ for all μ in $\mathbf{C}$ and y in A. As in Theorem 5.8, this shows that $ay = ya$ for all y in A as desired.

The radical of a Banach algebra has other characterizations which are needed for the next theorem. A *representation* π of a Banach algebra A is a homomorphism from A into $B(X)$, the Banach algebra of bounded linear operators on a Banach space X. The representation π is called *irre-*

ducible provided that $\pi(x)$ has no non-trivial invariant subspaces for every x in A. It is *faithful* if $\ker \pi = (0)$. The kernel of an irreducible representation π is called a *primitive* ideal of A and has the following intrinsic characterization: a two-sided ideal J is primitive if and only if there is a maximal modular left ideal L such that $J = \{a \in A : ax \in L \text{ for every } x \in A\}$. The radical of A can then be characterized as the intersection of all primitive ideals of A (See Rickart [73, pp. 54-55]). An algebra A is called *primitive* if (0) is a primitive ideal, *semi-simple* if $\mathrm{Rad}(A) = (0)$, and *radical* if $\mathrm{Rad}(A) = A$.

THEOREM 5.10. [Le Page]. *Suppose A is a semi-simple Banach algebra with identity and set $D_x y = xy - yx$. If one of the two following conditions holds, then A is commutative:*

(1) *For all x, y in A and $\varepsilon > 0$ there exists $M > 0$ such that $\| \exp(\lambda x) y \exp(-\lambda x) \| \leqslant M \exp(\varepsilon |\lambda|)$ for all λ in* $\mathbf{C}$.

(2) *For all x, y in A, $\| D_x^n y \|^{1/n} \to 0$ as $n \to \infty$.*

Proof. Conditions (1) and (2) are equivalent since $\exp(\lambda x) y \exp(-\lambda x) = \sum_{n=0}^{\infty} (\lambda^n/n) D_x^n y$ (See, for example Markushevich [56, vol. II, p. 259].) If A satisfies (1) or (2), so does every primitive quotient algebra $B = A/J$. But B can be represented faithfully as an irreducible algebra of bounded operators on a Banach space X. We wish to show that B is one-dimensional. Suppose, to the contrary, there is an operator U in B and an a in X such that $b = U(a)$ is not a scalar multiple of a. Set $D_T U = TU - UT$ and choose T in B such that $T(a) = 0$ and $T(b) = b$, which is possible by the strict transitivity of B (See Rickart [72, p. 60]).

Then $D_T U(a) = TU(a) - UT(a) = b$ and $D_T^n U(a) = b$ for all $n > 0$. Thus $\| D_T^n U \| \geqslant \| b \| / \| a \|$, which contradicts (2). We conclude that every irreducible representation of A is one dimensional, hence commutative. Then $\pi(xy - yx) = 0$ for every x, y in A and irreducible representation π. Since A is semi-simple, $xy - yx = 0$.

Donald Spicer [82] has observed that if every irreducible representation of A is one-dimensional and there exists $k > 0$ such that $\| x \| \leqslant k\rho(x)$ for all x in the commutator C of A, then A is commutative; and conversely. Indeed, since every irreducible representation of A/R is one-dimensional, then A/R is commutative and hence $C \subset \mathrm{Rad}(A)$. Therefore every commutator is quasi-nilpotent and it follows that $xy - yx = 0$ by the inequality.

THEOREM 5.11. [Le Page]. *If A is a complex Banach algebra with identity and $Ax^2 = Ax$ for every x in A, then A is commutative and semi-simple.*

Proof. To prove that A is semi-simple it suffices to show that 0 is the only quasi-nilpotent element of A. If x is quasi-nilpotent, there is a y in A such that $yx^2 = x$. So for every positive integer p, $y^p x^{p+1} = x$ or $\|x\|^{1/p} \leqslant \|y\| \; \|x\|^{1/p} \|x^p\|^{1/p} \to 0$ as $p \to \infty$. Thus $\|x\| = 0$ and $x = 0$. Next note that if $yx^2 = x$, yx is idempotent. Since $[x(yx-1)]^2 = x(yx^2-x)(yx-1) = 0$, $x(yx-1)$ is nilpotent, hence zero. Premultiplying by y yields $(yx)^2 - yx = 0$. Now every idempotent e in A is central: for any z in A, $ez(1-e)$ and $(1-e)ze$ are nilpotent, hence zero. Thus ez and ze are both eze.

Now the hypothesis of 5.11 will be fulfilled in every quotient and in particular every primitive quotient $B = A/J$ of A. If x is a non-zero element of B, there is a y in B such that yx is nonzero and central. Since a central element in an algebra of operators is multiplication by a scalar, x is left invertible. This implies $B \simeq \mathbf{C}$ and so every irreducible representation of A is one-dimensional. As in Theorem 5.10 we conclude that A is commutative.

J. Duncan and A. Tullo [36] have extended Theorem 5.11 to show that A must be finite-dimensional and in fact $A \approx \mathbf{C}^n$.

It is easily seen that the extended result of Duncan and Tullo fails if A is not required to have identity; e.g., let A be the space of complex sequences which are eventually zero.

THEOREM 5.12. [Le Page]. *Suppose A is a Banach algebra with identity and $ax - xa$ is quasi-nilpotent for every x in A. Then $ax - xa \in \mathrm{Rad}(A)$ for all x in A.*

Proof. We proceed as in the proof of Theorem 5.10, first noting that the hypothesis on a holds for its canonical image in any quotient of A. Let $B = A/J$ be any primitive quotient, considering B as an irreducible algebra of operators on X, and let U be the image of a in B. We show U is in the center of B by showing that it is multiplication by a scalar. This is immediate if $\dim X = 1$, so assume to the contrary that $\dim X > 1$ and that there exists $b \in X$ such that $b' = U(b)$ is not a scalar multiple of b. Choose T in B such that $T(b) = 0$ and $T(b') = b$ by the strict transitivity of B. Let $V = TU - UT$ so that $V(b) = b$, implying that $1 \in \sigma(V)$ and hence V fails to be quasi-nilpotent. Thus the image of $ax - xa$ is zero by every irreducible representation and consequently belongs to $\mathrm{Rad}(A)$.

We do not know if the assumption of an identity element can be deleted in (5.9)-(5.12). Also, it is open as to whether or not one needs completeness in (5.11) and (5.12).

Hirschfeld and Żelazko closed their influential paper on commutativity [47] with the following two conjectures:

Conjecture 1. If for every commutative subalgebra B of a complex Banach algebra A there is a constant k such that $\rho(x) \geqslant k \| x \|$ for every x in B, then A is commutative.

Conjecture 2. If A is a complex Banach algebra in which 0 is the only quasi-nilpotent and the spectral radius is continuous, then A is commutative.

B. Aupetit has established partial results in the direction of Conjecture 1 [10] and more recently has published a counterexample to Conjecture 2 [15]. In the description which follows the continuity of $x \to \sigma(x)$ refers to the Hausdorff metric on the compact subsets of **C**. A proof of the next result is given in [10, Theorem 1.1].

THEOREM 5.13. [Aupetit]. *If for every a in a complex Banach algebra A there is a constant $k > 0$ such that $\rho(x) \geqslant k \| x \|$ for every x in the closed subalgebra generated by a, then the function $x \to \sigma(x)$ is locally uniformly continuous on an open dense subset of A.*

Global uniform continuity of the spectrum, or even of the spectral radius, on A would imply commutativity modulo the radical as we shall see in the next section. The proof of Theorem 5.13 and the proofs published by Aupetit on commutativity and the spectral radius conditions have been based on potential theory and the use of subharmonic functions. These techniques render the proofs less computational, but also less elementary and accessible. They have however drawn analytical and topological considerations further into the picture and produced some very appealing theorems. For example in another partial result on Conjecture 1, conditions on the topological properties of the spectrum are used to obtain sufficiency for commutativity [10]:

THEOREM 5.14. *If for every x in A there is an a in A and $k > 0$ such that x lies in the closed subalgebra $C(a)$ generated by a, $\sigma(a)$ has no interior points and a finite number of holes, and $\rho(y) \geqslant k \| y \|$ for every $y \in C(a)$, then A is commutative.*

A surprisingly simple counterexample to Conjecture 2 was published by Aupetit in 1978 [15]. He makes extensive use of results obtained by Ackermans [1] in which the Gelfand representation for a commutative Banach algebra B is lifted to the matrix algebra with entries in B. Let U be the open unit disk in $\mathbf{C}$ and B the commutative Banach algebra of continuous functions on $\bar{U} \times \bar{U}$ which are holomorphic in $U \times U$. In the algebra of 2×2 matrices with entries in B define the norm by

$$\left\| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right\| = \max \{ \| a \| + \| b \|, \| c \| + \| d \| \} .$$

Let A be the closed noncommutative subalgebra with 1 formed by the matrices

$$m = \begin{pmatrix} f(z_1, z_2) & g(z_1, z_2) \\ (z_1 + z_2) Tg(z_1, z_2) & Tf(z_1, z_2) \end{pmatrix} \text{ where } f, g \in B ,$$

and T is the isometric automorphism of B defined by $Tf(z_1, z_2) = f(z_2, z_1)$. According to Ackermans [1, Th. 3.1, 3.2] the spectrum is continuous on A. If $m \in A$ is quasi-nilpotent then again by [1, Th. 2.2] $\{0\} = \bigcup_{\phi \notin \hat{B}} \sigma(\tilde{\phi}(m))$ where $\hat{B}$ is the set of multiplicative linear functionals on A and

$$\tilde{\phi}(m) = \begin{pmatrix} \phi(f(z_1, z_2)) & \phi(g(z_1, z_2)) \\ \phi((z_1 + z_2) Tg(z_1, z_2)) & \phi(Tf(z_1, z_2)) \end{pmatrix}$$

Thus $\tilde{\phi}(m)$ is quasi-nilpotent for each $\phi \in \hat{B}$ and its square is zero by the Cayley-Hamilton Theorem. Since B is semi-simple, $m^2 = 0$. In particular $f(z_1, z_2)^2 + (z_1 + z_2) g(z_1, z_2) g(z_2, z_1) \equiv 0$, which in turn implies that $f(z_1, z_2) \equiv g(z_1, z_2) \equiv 0$. Hence $m = 0$ so there are no nonzero quasi-nilpotents in A.

6. Commutativity and the spectral radius

We now consider some weaker conditions on the spectral radius which influence the commutativity of a Banach algebra. Two familiar properties of the norm, subadditivity and submultiplicativity, are also satisfied by the spectral radius on *commuting* elements. Does the imposition of these properties on the whole algebra then imply commutativity? Because