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Autor(en): **Nachbin, Leopoldo**

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WHY HOLOMORPHY IN INFINITE DIMENSIONS?

by Leopoldo NACHBIN

> Ein Mathematiker, der nicht etwas Poet ist, wird nie ein vollkommener Mathematiker
>
> KARL WEIERSTRASS

1. INTRODUCTION

The study of holomorphic functions in infinite dimensions is an objective as old in Mathematics as Functional Analysis, and as the idea of systems with an infinite number of degrees of freedom in Mechanics. It dates back to the end of the last century. The simple language of normed spaces and of topological vector spaces became a routine, as a suitable form of linear algebra in infinite dimensions to be used in Analysis, Geometry and applications. Thereafter, the theory of holomorphic mappings in infinite dimensions was properly developed as a confluence of ideas and methods originating mostly from several complex variables, manifold theory and Functional Analysis. Independently of that, users of sophisticated mathematical methods in applications have employed and furthered holomorphy in infinite dimensions, in fields such as Mathematical Physics and Electrical Engineering. The present expository article was written by aiming at the non-specialists, more exactly, at the non-mathematicians. We will use Weierstrass' definition as a model for the general case.

2. SOME CLASSICAL MOTIVATIONS

EXAMPLE 1: SPECTRAL THEORY. If $Z: E \to E$ is a linear operator on the complex vector space E of finite dimension $n = 1, 2, \ldots$, the homogeneous linear equation $Z(x) = \lambda x$ has at least some solution $x \in E$, $x \neq 0$ for at least some $\lambda \in \mathbf{C}$. Equivalently, there is at least some $\lambda \in \mathbf{C}$ such that $\lambda I - Z$ is not invertible in the algebra $\mathscr{L}(E; E)$ of all linear operators on E, where I is the identity mapping of E; the set of all such λ has at most n elements. This fact is proved by noticing that $\lambda I - Z$ is not invertible if and only if, by taking determinants, the algebraic equation det $(\lambda I - Z) = 0$ is satisfied by λ. Now, we notice that det $(\lambda I - Z)$ is a polynomial in λ whose leading term is λ^n, hence of degree n. By the so-called fundamental theorem of Algebra, that algebraic equation has at least a solution $\lambda \in \mathbf{C}$; it is clear that it has at most n such solutions. This result is one of the starting points of Spectral Theory. A more general form of it is the following one. Let now $Z: E \to E$ be a continuous linear operator on the complex Banach space $E \neq 0$. There is at least some $\lambda \in \mathbf{C}$ such that $\lambda I - Z$ is not invertible in the Banach algebra $\mathscr{L}(E; E)$ of all continuous linear operators on E, where I is the identity mapping of E. This result is no longer proved as in the finite dimensional situation, as we no longer have analogues of determinant theory and of the fundamental theorem of Algebra, as formerly. Since there is no difference in terms of difficulty in the exposition, we will explain this aspect in the more general language of Banach algebras. Let then A be a complex Banach algebra with unit $I \neq 0$; thus A is a Banach space and at the same time an algebra for which $\| XY \| \leqslant \| X \| \cdot \| Y \|$ if $X, Y \in A$, and $\| I \| = 1$. [For instance, if $E \neq 0$ is a complex Banach space, then $\mathscr{L}(E; E)$ is a complex Banach algebra with unit $I \neq 0$ in a natural way.] The spectrum spt (Z) of $Z \in A$ is the set of all $\lambda \in \mathbf{C}$ such that $\lambda I - Z$ is not invertible in A. The Gelfand-Mazur theorem states that spt (Z) is always nonvoid; it is clear that it is compact in $\mathbf{C}$. How can we prove such a result without analogues of determinant theory and of a fundamental theorem of Algebra? Surprisingly enough at first sight, this is accomplished through a seemingly isolated result in Complex Analysis, known as the Liouville theorem: if an entire complex valued function of a complex variable is bounded, then it must be a constant. As a matter of fact, it is immediately pointed out in Complex Analysis courses that a possible application of Liouville theorem is to a proof of the fundamental theorem of Algebra. Coming back to the Gelfand-Mazur theorem, its short but smart proof goes as follows. Assume that Z has a void spectrum. The vector valued function $\lambda \in \mathbf{C} \mapsto (\lambda I - Z)^{-1} \in A$ of a complex variable is entire, and it tends to zero at infinity. Thus the function in question must be a constant, by Liouville theorem, once it is entire and bounded; actually it must be identically zero as it is a constant and tends to zero at infinity. However, this is an absurdity as no inverse in A can be zero. The above proof calls for the need of a vector valued Liouville theorem of a complex variable, which not only is true but may be proved as easily as the scalar valued one. It is true that we may bypass the vector valued Liouville theorem by arguing as follows. For every continuous linear form φ on A, the scalar valued function $\lambda \in \mathbf{C} \mapsto \varphi\,[(\lambda I - Z)^{-1}] \in \mathbf{C}$ of a complex variable is entire, and it tends to zero at infinity. By the classical Liouville theorem, this function is identically zero for every such φ. By the Hahn-Banach theorem, if $X \in A$ satisfies $\varphi(X) = 0$ for every such φ, then $X = 0$. Thus $(\lambda I - Z)^{-1} = 0$ for all $\lambda \in \mathbf{C}$. However, this is an absurdity as no inverse in A can be zero. This equally nice proof of the Gelfand-Mazur theorem, via the classical Liouville theorem plus (the unnecessary use of) the Hahn-Banach theorem is like a good dessert whose recipe the cook does not tell us!... See Example 1 in Section 3 below.

Example 2: Operational calculus. As in Example 1, we could consider the Banach algebra $\mathscr{L}(E; E)$ associated to a complex Banach space $E \neq 0$. Since there is no difference in terms of difficulty in the exposition, we will explain this aspect in the more general language of Banach algebras. Let then A be as in Example 1. If $f\colon \mathbf{C} \to \mathbf{C}$ is entire, we may consider its Taylor series

$$f(z) = \sum_{m=0}^{\infty} a_m (z - \xi)^m$$

about $\xi \in \mathbf{C}$, for any $z \in \mathbf{C}$, where $a_m = f^{(m)}(\xi)/m!$ for $m \in \mathbf{N}$. It is natural to define

$$f(Z) = \sum_{m=0}^{\infty} a_m (Z - \xi I)^m$$

for any $Z \in A$. It is easily checked that this definition makes sense, once $|a_m|^{1/m} \to 0$ as $m \to \infty$; and that $f(Z) \in A$ does not depend on the choice of ξ. Since the function $z \in \mathbf{C} \mapsto f(z) \in \mathbf{C}$ is entire, we would like to have a terminology allowing us to assert that the mapping $Z \in A \mapsto f(Z) \in A$ is entire too. For a change, consider now the nonvoid open subset $A^* \subset A$ formed by the invertible elements of A, and the nonvoid open subset $\mathbf{C}^* \subset \mathbf{C}$ formed by the nonzero elements of $\mathbf{C}$. Since the function $z \in \mathbf{C}^* \mapsto 1/z \in \mathbf{C}$ is holomorphic, we would like to have a terminology allowing us to assert that the mapping $Z \in A^* \mapsto Z^{-1} \in A$ is holomorphic too. More generally, let $\mathscr{H}(U; \mathbf{C})$ denote the algebra of all holomorphic functions $f\colon U \to \mathbf{C}$, where $U \subset \mathbf{C}$ is open nonvoid. If $f \in \mathscr{H}(\mathrm{U}; \mathbf{C})$ and J is an oriented, rectifiable Jordan contour (formed by an exterior, counterclockwise oriented, rectifiable Jordan curve and a finite number of interior, mutually exterior, clockwise oriented, rectifiable Jordan curves) fitted in U, we may consider the Cauchy integral

$$f(z) = \frac{1}{2\pi i} \int_{\lambda \in J} \frac{f(\lambda)}{\lambda - z} \, d\lambda$$

for any $z \in U$, provided z is surrounded by J. It is natural to define

$$f(Z) = \frac{1}{2\pi i} \int_{\lambda \in J} f(\lambda)(\lambda I - Z)^{-1} \, d\lambda$$

for any $Z \in A$ such that spt $(Z) \subset U$, provided spt (Z) is surrounded by J. It is easily checked that $f(Z) \in A$ does not depend on the choice of such J. The two previous cases are subsumed by the present one. Consider now the nonvoid open subset A^U of A formed by all $Z \in A$ such that spt $(Z) \subset U$. Since the function $z \in U \mapsto f(z) \in \mathbf{C}$ is holomorphic, we would like to have a terminology allowing us to assert that the mapping $Z \in A^U \mapsto f(Z) \in A$ is holomorphic too. This is indeed the case with the natural definition of holomorphic mappings between normed spaces. See Example 2 in Section 3 below.

EXAMPLE 3: ORDINARY DIFFERENTIAL EQUATIONS. Consider an open subset $U \subset \mathbf{C}^2$ containing $(z_0, w_0) \in \mathbf{C}^2$ and a holomorphic function $f : U \to \mathbf{C}$. The classical existence and uniqueness theorem concerning the ordinary differential equation $w' = f(z, w)$ reads as follows. If $\delta \in \mathbf{R}$, $\delta > 0$, let $B_\delta(z_0)$ be the set of all $z \in \mathbf{C}$ satisfying $|z - z_0| < \delta$. For some such δ, there is a holomorphic function $g : B_\delta(z_0) \to \mathbf{C}$ such that $g(z_0) = w_0$, $(z, g(z)) \in U$ and $g'(z) = f[z, g(z)]$ if $z \in B_\delta(z_0)$. Moreover, if for some such δ we have two holomorphic functions $g_j : B_\delta(z_0) \to \mathbf{C}$ such that $g_j(z_0) = w_0$, $(z, g_j(z)) \in U$ and $g'_j(z) = f[z, g_j(z)]$ if $z \in B_\delta(z_0)$, where $j = 1, 2$, then $g_1(z) = g_2(z)$ for $z \in B_\delta(z)$. By keeping z_0, w_0 fixed, we would like to have a terminology allowing us to assert that the solution $w = g(z)$ passing through (z_0, w_0) of $w' = f(z, w)$ varies holomorphically with $f(z, w)$. This is done in elementary courses as follows. Consider an open subset $V \subset \mathbf{C}^3$ containing $(z_0, w_0, \lambda_0) \in \mathbf{C}^3$ and a holomorphic function $f : V \to \mathbf{C}$. The classical theorem concerning the ordinary differential equation $w' = f(z, w, \lambda)$ depending on the parameter λ reads as follows. For some $\delta \in \mathbf{R}$, $\delta > 0$, there is a holomorphic function $g : B_\delta(z_0) \times B_\delta(\lambda_0) \to \mathbf{C}$ such that $g(z_0, \lambda) = w_0$ if $\lambda \in B_\delta(\lambda_0)$, $(z, g(z, \lambda), \lambda) \in V$ and $g'_z(z, \lambda) = f[z, g(z, \lambda), \lambda]$ if $z \in B_\delta(z_0)$, $\lambda \in B_\delta(\lambda_0)$. Likewise if we have several parameters. We then say that, if an ordinary differential equation depends holomorphically on the variable, the unknown and the parameters, then its solution through a fixed point depends holomorphically on the variable and the parameters. See Example 3 in Section 3 below.

EXAMPLE 4: IMPLICIT FUNCTION THEOREM. Consider an open subset $U \subset \mathbf{C}^2$ containing $(z_0, w_0) \in \mathbf{C}^2$ and a holomorphic function $f : U \to \mathbf{C}$. The classical existence and uniqueness theorem concerning the implicit function equation $f(z, w) = 0$ reads as follows. Assume that $f(z_0, w_0) = 0$ and $f'_w(z_0, w_0) \neq 0$. For some $\delta \in \mathbf{R}$, $\delta > 0$, there is a holomorphic function $g : B_\delta(z_0) \to \mathbf{C}$ such that $g(z_0) = w_0$, $(z, g(z)) \in U$ and $f[z, g(z)] = 0$ if $z \in B_\delta(z_0)$. Moreover, if for some such δ, we have two holomorphic functions $g_j : B_\delta(z_0) \to \mathbf{C}$ such that $g_j(z_0) = w_o$, $(z, g_j(z)) \in U$ and $f[z, g_j(z)] = 0$ if $z \in B_\delta(z_0)$, where $j = 1, 2$, then $g_1(z) = g_2(z)$ for $z \in B_\delta(z_0)$. By keeping z_0, w_0 fixed, we would like to have a terminology allowing us to assert that the solution $w = g(z)$ passing through (z_0, w_0) of $f(z, w) = 0$ varies holomorphically with $f(z, w)$. This is done in elementary courses as follows. Consider an open subset $V \subset \mathbf{C}^3$ containing $(z_0, w_0, \lambda_0) \in \mathbf{C}^3$ and a holomorphic function $f : V \to \mathbf{C}$. The classical theorem concerning the implicit function equation $f(z, w, \lambda) = 0$ depending on the parameter λ reads as follows. Assume that $f(z_0, w_0, \lambda) = 0$ and $f'_w(z_0, w_0, \lambda) \neq 0$ if $(z_0, w_0, \lambda) \in V$. For some $\delta \in \mathbf{R}$, $\delta > 0$, there is a holomorphic function $g : B_\delta(z_0) \times B_\delta(\lambda_0) \to \mathbf{C}$ such that $g(z_0, \lambda) = w_0$ if $\lambda \in B_\delta(\lambda_0)$, $(z, g(z, \lambda), \lambda) \in V$ and $f[z, g(z, \lambda), \lambda] = 0$ if $z \in B_\delta(z_0)$, $\lambda \in B_\delta(\lambda_0)$. Likewise if we have several parameters. We then say that, if an implicit function equation depends holomorphically on the variable, the unknown and the parameters, then its solution through a fixed point depends holomorphically on the variable and the parameters. See Example 4 of Section 3 below.

3. HOLOMORPHIC MAPPINGS

The topological vector spaces language is becoming a routine method of expression in Mathematics and certain of its applications, say to Mathematical Physics, Engineering and Economics. Our standard references are [6], [13], [15], [17], [31] and [32].

Let us recall that a complex topological vector space E is a vector space which at the same time is a topological space, such that the vector space operations $(x, y) \in E \times E \mapsto x + y \in E$ and $(\lambda, x) \in \mathbf{C} \times E \mapsto \lambda x \in E$ are continuous. A seminorm on a complex vector space E is a function $\alpha : E \mapsto \mathbf{R}_+$ such that $\alpha(x_1 + x_2) \leqslant \alpha(x_1) + \alpha(x_2)$ and $\alpha(\lambda x) = |\lambda| \cdot \alpha(x)$ $x_1, x_2, x \in E$, $\lambda \in \mathbf{C}$. We denote by $CS(E)$ the set of all continuous seminorms on a topological vector space E. If Γ is a nonvoid set of seminorms on a vector space E, we define the associated topology $\mathcal{T}_\Gamma$ on E by saying that $X \subset E$ is open if, whenever $x \in X$, there are $\alpha_1, ..., \alpha_n \in \Gamma$, $\varepsilon > 0$ for which $t \in E$, $\alpha_i(t-x) < \varepsilon$ for $i = 1, ..., n$ imply that $t \in X$. Then E is a topological vector space if endowed with $\mathscr{I}_\Gamma$. Which topological vector spaces do we get this way from arbitrary E and Γ? Well, $X \subset E$ is convex if, whenever $x_0, x_1 \in X$, $\lambda \in \mathbf{R}$, $0 \leqslant \lambda \leqslant 1$, then $(1-\lambda)x_0 + \lambda x_1 \in X$. A topological vector space E is locally convex if the convex neighborhoods of every $x \in E$ form a basis of neighborhoods of x; it suffices to check that at one point, say the origin. If Γ is a nonvoid set of seminorms on a vector space E, then E endowed with $\mathscr{I}_\Gamma$ is locally convex and $\Gamma \subset CS(E)$. Conversely, if E is a locally convex space, its topology $\mathscr{I}$ is associated to $\Gamma = CS(E)$, that is $\mathscr{I} = \mathscr{I}_\Gamma$. Hence locally convex spaces are just topological vector spaces whose topologies are defined by nonvoid sets of seminorms. There are basic results, such as the Hahn-Banach theorem, that are valid for locally convex spaces, but not necessarily for topological vector spaces. Fortunately, most topological vector spaces that we encounter are locally convex and have their topologies associated to sets Γ at sight. It is true that there are topological vector spaces that are not locally convex but are used, say in probability theory, typically $L^p(\mu)$ of a measure μ with $0 \leqslant p < 1$.

Fix the complex locally convex spaces E, F.

If $m = 1, 2, ...$, let $\mathscr{L}_a(^mE; F)$ be the vector space of all m-linear mappings of the cartesian power E^m to F; and $\mathscr{L}_{as}(^mE; F)$ be the vector subspace of all symmetric such mappings. Here (and in the sequel) the index "a" stands for "algebraic", continuity not being assumed. Let $\mathscr{L}(^mE; F)$ and $\mathscr{L}_s(^mE; F)$ be the vector subspaces of those $A \in \mathscr{L}_a(^mE; F)$ and $A \in \mathscr{L}_{as}(^mE; F)$ that are continuous, respectively. If $m = 0$, we set $\mathscr{L}_a(^0E; F) = \mathscr{L}_{as}(^0E; F) = \mathscr{L}(^0E; F) = \mathscr{L}_s(^0E; F) = F$.

Letting $A \in \mathscr{L}_a(^mE; F)$, $x \in E$, write $Ax^m = A(x, ..., x)$ if $m = 1, 2, ...$; and $Ax^0 = A$ if $m = 0$. To every such A, associate the mapping $\hat{A} : E \mapsto F$ defined by $\hat{A}(x) = Ax^m$ if $x \in E$. Call $\hat{A}$ the m-homogeneous polynomial associated to A. Denote by $\mathscr{P}_a(^mE; F)$ the vector space of all m-homogeneous polynomials of E to F associated to all $A \in \mathscr{L}_a(^mE; F)$; and by $\mathscr{P}(^mE; F)$ the vector subspace of all continuous such polynomials. The linear mappings $A \in \mathscr{L}_a(^mE; F) \mapsto \hat{A} \in \mathscr{P}_a(^mE; F)$ and $A \in \mathscr{L}(^mE; F) \mapsto \hat{A} \in \mathscr{P}(^mE; F)$ are surjective. Moreover, the linear mappings $A \in \mathscr{L}_{as}(^mE; F) \mapsto \hat{A} \in \mathscr{P}_a(^mE; F)$ and $A \in \mathscr{L}_s(^mE; F) \mapsto \hat{A} \in \mathscr{P}(^mE; F)$ are bijective.

Let $U \subset E$ be open and nonvoid. We say that $f : U \to F$ is holomorphic if, corresponding to every $\xi \in U$, there are Taylor coefficients $A_m \in \mathscr{L}_s(^mE; F)$

for $m = 0, 1, \ldots$ such that, for every $\beta \in CS(F)$, there is a neighborhood V of ξ in U for which

$$\lim_{m \to \infty} \beta \left[f(x) - \sum_{k=0}^{m} A_k (x - \xi)^k \right] = 0$$

uniformly for $x \in V$. Let $\mathscr{H}(U; F)$ be the vector space of all holomorphic mappings of U to F. If F is a normed space, the definition that f is holomorphic means that, corresponding to every $\xi \in U$, there are $A_m \in \mathscr{L}_s({}^mE; F)$ for $m = 0, 1, \ldots$ such that

$$f(x) = \sum_{m=0}^{\infty} A_m (x - \xi)^m,$$

convergence being uniform for x in some neighborhood V of ξ in U. In general, the definition must be given as we phrased it.

If F is a Hausdorff space, the sequence (A_m) of Taylor coefficients of $f \in \mathscr{H}(U; F)$ at $\xi \in U$ is unique. Then

$$f(x) \cong \sum_{m=0}^{\infty} A_m (x - \xi)^m$$

is called the Taylor series of f at ξ, where $x \in U$. We define the m-differentials of f at ξ by

$$d^m f(\xi) = m! A_m, \quad \hat{d}^m f(\xi) = m! \hat{A}_m$$

considered as elements of $\mathscr{L}_s({}^mE; F)$ and $\rho({}^mE; F)$ respectively, for $m = 0, 1, \ldots$. The Taylor series of f at ξ becomes

$$f(x) \cong \sum_{m=0}^{\infty} \frac{1}{m!} d^m f(\xi)(x - \xi)^m$$
$$\cong \sum_{m=0}^{\infty} \frac{1}{m!} \hat{d}^m f(\xi)(x - \xi).$$

Example 1: Spectral theory. If F is a complex normed space and $f \in \mathscr{H}(\mathbf{C}; F)$ is bounded, the vector valued Liouville theorem asserts that f is a constant; this is proved exactly in the same way as when $F = \mathbf{C}$, that is, as in the classical Liouville theorem. This simple result was used in the proof of the Gelfand-Mazur theorem as given in Example 1 of Section 2. More generally, if E, F are complex locally convex spaces, F is a Hausdorff space, and $f \in \mathscr{H}(E; F)$ is bounded, that is $f(E)$ is bounded in F, then f is a constant; this is proved by a simple reduction to the case when $E = \mathbf{C}$ and F is a normed space. We recall that $Y \subset F$ is bounded in F if, for every neighborhood V of 0 in F, there is $\lambda \in \mathbf{C}$ such that $Y \subset \lambda V$.

EXAMPLE 2: OPERATIONAL CALCULUS. In the notation and terminology of Example 2 of Section 2, once $f \in \mathscr{H}(U; \mathbf{C})$ is fixed, the mapping $Z \in A^U \to f(Z) \in A$ is indeed holomorphic. All this becomes a more venturous enterprise in the more general case when, in the notation of Example 2 of Section 2, E is a locally convex space, or A is a locally convex algebra.

In order to reconsider Examples 3 and 4 of Section 2, we need to describe an important example of locally convex spaces, namely $\mathscr{H}(K; \mathbf{C})$, the space of germs of complex valued functions that are holomorphic around a fixed nonvoid compact subset K of $\mathbf{C}^n$. This example became a routine in Complex Analysis, Functional Analysis and applications. However, what happened historically may be described as follows. Fantappiè and others studied a lot the so-called analytic functionals, that is functions whose variable is an analytic (holomorphic) functions. Yet Fantappiè did not know how to introduce and use a natural topology on the spaces of holomorphic functions that he considered. Accordingly, he had to bypass this handicap to a certain extent. When Laurent Schwartz developed the theory of distributions, he naturally considered inductive (direct) limits. The most basic example of them in his theory is the following one. Once a nonvoid open subset $U \subset \mathbf{R}^n$ is fixed, the vector space $\mathscr{D}(U; \mathbf{C})$ of all infinitely differentiable complex valued functions on U with compact supports contained in U is to be looked upon as an inductive limit of the vector space $\mathscr{D}(U; \mathbf{C})$ of all such functions with supports contained in K, for any compact subset $K \subset U$. Next Dieudonné and Schwartz wrote an article on basic aspects of inductive limits of locally convex spaces. This led Dias, Grothendieck and Köthe simultaneously to define the natural topology on $\mathscr{H}(K; \mathbf{C})$ as follows.

Fix then a nonvoid compact subset $K \subset \mathbf{C}^n$ and consider the union

$$\mathscr{H}[K; \mathbf{C}] = \bigcup_{U \supset K} \mathscr{H}(U; \mathbf{C})$$

where U varies over all open subsets of $\mathbf{C}^n$ containing K. Define an equivalence relation modulo K on that union by considering $f_i : U_i \to \mathbf{C}$ $(i = 1, 2)$ as equivalent if $U_i \subset \mathbf{C}^n$ is open containing K and $f_i \in \mathscr{H}(U_i; \mathbf{C})$, the set of points $x \in U_1 \cap U_2$ satisfying $f(x_1) = f(x_2)$ being a neighborhood of K in $\mathbf{C}^n$. Each equivalence class of $\mathscr{H}[K; \mathbf{C}]$ modulo such an equivalence relation is called a germ of holomorphic function around K. If $f \in \mathscr{H}[K; \mathbf{C}]$, we denote by $\tilde{f}_K$, or simply $\tilde{f}$, its equivalence class, that is, its germ modulo K. Call $\mathscr{H}(K; \mathbf{C})$ the quotient space of $\mathscr{H}[K; \mathbf{C}]$ modulo that equivalence relation. Then $\mathscr{H}(K; \mathbf{C})$ is a vector space in a unique way so that every mapping $f \in \mathscr{H}(U; \mathbf{C}) \mapsto \tilde{f}_K \in \mathscr{H}(K; \mathbf{C})$ is linear, where $U \subset \mathbf{C}^n$ is open containing K. Denote by $\mathscr{H}_B(U; \mathbf{C})$ the Banach space of all $f \in \mathscr{H}(U; \mathbf{C})$ that are bounded on U, where $\mathscr{H}_B(U; \mathbf{C})$ is endowed with the supremum norm. The natural topology on $\mathscr{H}(K; \mathbf{C})$ is defined by the following inductive limit procedure: it is the largest locally convex topology on $\mathscr{H}(K; \mathbf{C})$ such that each linear mapping $f \in \mathscr{H}_B(U; \mathbf{C}) \mapsto \tilde{f}_K \in \mathscr{H}(K; \mathbf{C})$ is continuous, for every open subset $U \subset \mathbf{C}^n$ containing K. We could also use an alternative form of this definition. The natural topology used on $\mathscr{H}(U; \mathbf{C})$ is the so-called compact-open topology. Then the same natural topology on $\mathscr{H}(K; \mathbf{C})$ may be defined by the following inductive limit procedure: it is the largest locally convex topology on $\mathscr{H}(K; \mathbf{C})$ such that each linear mapping $f \in \mathscr{H}(U; \mathbf{C}) \mapsto \tilde{f}_K \in \mathscr{H}(K; \mathbf{C})$ is continuous, for every open subset $U \subset \mathbf{C}^n$ containing K. If $K = \{z\}$ is reduced to a point $z = (z_1, ..., z_n) \in \mathbf{C}^n$, we write $\mathscr{H}(z; \mathbf{C}) = \mathscr{H}(z_1, ..., z_n; \mathbf{C})$ for $\mathscr{H}(\{z\}; \mathbf{C})$.

Example 3: Ordinary differential equations. Let us resume notation and terminology of Example 3 of Section 2. The classical existence and uniqueness theorem for ordinary differential equations allows us to associate to the germ $\tilde{f} \in \mathscr{H}(z_0, w_0; \mathbf{C})$ of $f \in \mathscr{H}(U; \mathbf{C})$ at (z_0, w_0) the germ $\tilde{g} \in \mathscr{H}(z_0; \mathbf{C})$ of $g \in \mathscr{H}(B_\delta(z_0); \mathbf{C})$ at z_0. It can be proved that the mapping $\tilde{f} \in \mathscr{H}(z_0, w_0; \mathbf{C}) \mapsto \tilde{g} \in \mathscr{H}(z_0; \mathbf{C})$ is holomorphic. It is really in this simple way that we should state that the solution passing through $(z_0; w_0)$ depends holomorphically on the differential equation. We see now how much exposition is needed to express that result in such a simple form. That is why we bypass such a language problem and state the result in the weaker classical form involving parameters; as a matter of fact, this is enough for certain purposes.

Example 4: Implicit function theorem. Let us resume notation and terminology of Example 4 of Section 2. Let $\mathscr{E}$ be the vector subspace of $\mathscr{H}(z_0, w_0; \mathbf{C})$ formed by all germs $\tilde{f} \in \mathscr{H}(z_0, w_0; \mathbf{C})$ of $f \in \mathscr{H}(U; \mathbf{C})$ at (z_0, w_0) satisfying $f(z_0, w_0) = 0$. Let $\mathscr{U}$ be the nonvoid open subset of $\mathscr{E}$ formed by those germs $\tilde{f}$ which, in addition to the above conditions, satisfy $f'_w(z_0, w_0) \neq 0$. The classical existence and uniqueness theorem for implicit function equations allows us to associate to the germ $\tilde{f} \in \mathscr{U}$ of $f \in \mathscr{H}(U; \mathbf{C})$ at (z_0, w_0) satisfying $f(z_0, w_0) = 0$, $f'_w(z_0, w_0) \neq 0$, the germ $\tilde{g} \in \mathscr{H}(z_0, \mathbf{C})$ of $g \in \mathscr{H}\left(B_\delta(z_0); \mathbf{C}\right)$ at z_0. It can be proved that the mapping $\tilde{f} \in \mathscr{U} \mapsto \tilde{g} \in \mathscr{H}(z_0; \mathbf{C})$ is holomorphic. We may repeat here some comments which are analogous to those made at the end of the above Example 3.

4. Concluding remarks

This article was written to attract prospective users in applications of holomorphy in infinite dimensions.

I have tried to illustrate through four very simple, classical examples, how the concept of holomorphic mappings in infinite dimensions comes up naturally in Analysis. The difference between Examples 1 and 2 on one side, and Examples 3 and 4 on the other side is striking: The first two examples seem very straightforward, while the last two examples look more sophisticated. However, sophistication in Mathematics is a matter of lack of habit; I personally am by now so used to dealing with germs of holomorphic functions that I no longer think of the last two examples as being sophisticated at all. Moreover, dealing long enough with any mathematical concept, particularly in applying it, leads to the development of a sort of intuition in that respect.

In 1963, I had my first opportunity of visiting Warsaw, and of talking leisurely to Mazur. I then played a little bit the role of a newspaper reporter and asked him if he, Banach and other members of the Polish group that developed Banach space theory, had specific applications in mind. Mazur answered, without any surprise to me as a mathematician, that the Polish group was guided by a conscience of the importance of Banach spaces in Mathematics proper. We witness nowadays how Banach spaces methods and results spread out in Mathematics and its applications. More accurately, Banach spaces have even been superseded by locally convex spaces for many of such goals. Psychologically, it is interesting to notice that the concept of a Banach space was also emphasized by Norbert Wiener; however Banach had a guiding idea to develop a fruitful theory, but that does not seen to be the case of Wiener in this particular instance. Likewise, the group that is developing holomorphy in infinite dimensions has been guided by a feeling of its possible interest in Mathematics proper. A promising direction of research at present seems to be the study of holomorphy linked to nuclearity in the sense of Grothendieck; interesting results in this direction have already been obtained, mainly by Boland and Dineen, but many such useful methods are to be expected in this area. Holomorphy in infinite dimensions is being used in Mathematical Physics, say in studying Fock spaces; and in Electrical Engineering through ideas originated from Volterra. However, the ties between the existing theory, or the theory to be developed, and its possible applications, are still loose, the reason being a lack of suitable interplay between mathematicians and users.

5. Some bibliographical references

The following references consist exclusively of some expository texts, and the proceedings of meetings. The readers should be able to trace back further information through them, concerning the various directions in which holomorphy in infinite dimensions branched off and is used. May I cite Kiselman in [16] below. He describes a problem in finite dimensions which was one of his motivations for the use of holomorphy in infinite dimensions; the problem has to do with the determination of the polynomially convex porters of a continuous linear form on $\mathscr{H}(\mathbf{C}^n; \mathbf{C})$ from the knowledge of its nonlinear Fourier-Borel transform. Actually, when I told Kiselman that I was preparing an article of motivation like the present one, he gladly wrote his article in [16] below, and suggested that the complete title of my article should be "Why Holomorphy in Infinite Dimensions? Why not?" According to an oral communication that I got from Dieudonné, one of the first authors to deal with holomorphy in infinite dimensions was D. Hilbert, in his article "Wesen und Ziele einer Analysis der unendlichvielen unabhägigen Variablen", *Rendiconti del Circolo Matematico di Palermo 27*, 1909, 59-74, or *Gesammelte Abhandlungen III*, 56-72.

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Leopoldo Nachbin

Department of Mathematics
University of Rochester
Rochester, New York 14627 USA

Departamento de Matemática Pura
Universidade Federal do Rio de Janeiro
Caixa postal 1835
21910 Rio de Janeiro, RJ, Brasil

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