§5. Hodge theory for $H^n(\Gamma; \rho, V)$, FROM THE VARIATION OF HODGE STRUCTURE

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$P^+K_{\mathbf{C}}$)-invariant subspace of V, and (iv) from the fact that $\mathfrak{p}^+F_0^r \subset F_0^{r-1}$ (resp. $\mathfrak{p}^-\bar{F}_0^s \subset \bar{F}_0^{s-1}$); both assertions follow from (1.7, i) and (4.9).

The flat polarization (4.6, d) ((4.5, v) in the real case) is provided by the admissible inner product T (1.9). Let $\mathfrak{C}_0$ denote the Weil operator of (4.8). Then

$$(4.11) \qquad T(v, w) = \bar{\beta}(\mathfrak{C}_0 v, w) \qquad \text{if we put} \qquad \bar{\beta}(v, w) = T(\mathfrak{C}_0^{-1} v, w).$$

We assert that $\bar{\beta}$ is G-invariant (with G acting by $\bar{\rho}$ on the second entry). For this, we need only apply (1.9) to see that

$$\bar{\beta}(\rho(X)v, w) + \bar{\beta}(v, \overline{\rho(X)}\, w) = 0$$

for all $X \in \mathfrak{g}_{\mathbf{C}}$, $v, w \in V$. (In the real case, we are displaying the self-contragredience of ρ.) That $\bar{\beta}$ determines a polarization now follows by homogeneity. This completes our verification.

Note that at $gx_0 \in M$,

$$(4.12) \qquad \begin{aligned} \bar{\beta}(\mathfrak{C}_{gx_0} v, w) &= \bar{\beta}(\rho(g)\mathfrak{C}_0\, \rho(g)^{-1} v, w) \\ &= \bar{\beta}(\mathfrak{C}_0\, \rho(g)^{-1} v, \overline{\rho(g)^{-1}}\, w) \\ &= T(\rho(g)^{-1} v, \overline{\rho(g)^{-1}}\, w), \end{aligned}$$

so the "Hodge metric" coincides with the one given in (2.8). Also, $\eta \in \mathscr{A}^n(S, \mathbf{V})$ takes its values in $\mathscr{H}^{p,q}$ if and only if $\tilde{\eta} \in \mathscr{A}^n(\Gamma\backslash G) \otimes H_0^{p,q}$.

§5. Hodge theory for $H^n(\Gamma; \rho, V)$, from the variation of Hodge structure

In this Section, we will review the general Hodge theory for locally constant sheaves $\mathbf{V}$ underlying polarizable variations of Hodge structure. After that, we will insert the construction of (4.9) into the general framework and draw special conclusions about this case. There are both local considerations and global results. The latter follow "automatically" only when S is compact, in which case they are due to Deligne (see [11, §§1-2]). The global results generalize to non-compact quotients of finite volume for $G = SL(2, \mathbf{R})$ [11, §§7, 12], and hopefully we will soon be able to handle $G = SU(n, 1)$. We should view the compact case as providing formal guidelines for a general theory.

Let $\mathbf{V}$ underlie a complex polarizable variation of Hodge structure of weight m on the compact Kähler manifold S, as in (4.6). Let

$$\mathscr{A}_{(2)}^{p,q}(S, \mathscr{H}^{r,s})$$

denote the space of square-summable C^∞ forms on S of type (p, q) with values in $\mathscr{H}^{r,s}$. Then

$$\mathscr{A}_{(2)}^{n}(S, \mathbf{V}) = \bigoplus_{\substack{p+q=n\\ r+s=m}} \mathscr{A}_{(2)}^{p,q}(S, \mathscr{H}^{r,s}). \tag{5.1}$$

As a consequence of (4.5, iv) and (4.6, c), it is easy to see that the operator d decomposes, under the splitting (5.1), into a sum of four operators, written $\partial', \bar{\partial}'$ ∇' and $\bar{\nabla}'$, which, in terms of the 4-fold gradation $(p, q; r, s)$, are respectively of degrees $(1, 0; 0, 0)$, $(0, 1; 0, 0)$, $(1, 0; -1, 1)$, and $(0, 1; 1, -1)$. We define

$$\begin{aligned} \mathfrak{D}' &= \partial' + \bar{\nabla}' \\ \mathfrak{D}'' &= \bar{\partial}' + \nabla'; \end{aligned} \tag{5.2}$$

the pairing of operators in (5.2) is done according to "total holomorphic" degree $p + r$. We have Laplacian operators for $\mathfrak{D}'$ and $\mathfrak{D}''$ as in (3.11). The following generalized Kähler identities hold:

(5.3) PROPOSITION (Deligne). $\Box_d = 2\Box_{\mathfrak{D}'} = 2\Box_{\mathfrak{D}''}$.

Proof. (See [11, §2]; the generalization to complex variations of Hodge structure is direct.)

Put

$$\mathscr{B}_{(2)}^{P,Q} = \bigoplus_{\substack{p+r=P\\ q+s=Q}} \mathscr{A}_{(2)}^{p,q}(S, \mathscr{H}^{r,s}). \tag{5.4}$$

In terms of this new bigrading, $\mathfrak{D}'$ is of bidegree $(1, 0)$ and $\mathfrak{D}''$ is of bidegree $(0, 1)$. As a consequence of (5.3), one obtains a decomposition of the harmonic forms into harmonic components of type (P, Q):

$$\mathscr{h}_{(2)}^{n}(S, \mathbf{V}) = \bigoplus_{P+Q=m+n} \mathscr{h}_{(2)}^{P,Q}, \tag{5.5}$$

with $\mathscr{h}_{(2)}^{P,Q} = \overline{\mathscr{h}_{(2)}^{Q,P}}$ in the real case. This decomposition passes to cohomology, as in (3.18), and thus we have

(5.6) THEOREM (Deligne). *Let* $\mathbf{V}$ *underlie a complex polarizable variation of Hodge structure of weight* m *on* S. *Then there is, for each* n, *an associated decomposition:*

$$\bar{H}^n_{(2)}(S, \mathbf{V}) = \bigoplus_{P+Q=m+n} \bar{H}^{P,Q}_{(2)}$$

If the variation of Hodge structure is real, then the above is a Hodge structure of weight $m+n$.

In order to work effectively with these decompositions, it is best to eliminate the C^∞ forms, and work only on the holomorphic level through the use of hypercohomology. Let $\Omega^\bullet_S(\mathbf{V})$ denote the holomorphic deRham complex with values in $\mathbf{V}$, with differential ∂. The following was given by Deligne:

(5.7) *Definition.* The *Hodge filtration* $\{F^r\Omega^\bullet_S(\mathbf{V})\}$ on $\Omega^\bullet_S(\mathbf{V})$ is given by

$$F^r\,\Omega^n_S(\mathbf{V}) = \Omega^n_S \otimes \mathscr{F}^{r-n};$$

because of (4.5, iv), $F^r\Omega^\bullet_S(\mathbf{V})$ is a sub-complex of $\Omega^\bullet_S(\mathbf{V})$.

The successive quotients

$$Gr^r_F\,\Omega^\bullet_S(\mathbf{V}) = F^r\,\Omega^\bullet_S(\mathbf{V})/F^{r+1}\Omega^\bullet_S(\mathbf{V})$$

have terms

(5.8) $$Gr^r_F\,\Omega^n_S(\mathbf{V}) = \Omega^n_S \otimes \mathscr{G}r^{r-n},$$

where $\mathscr{G}r^q = \mathscr{F}^q/\mathscr{F}^{q+1}$.

Then $\mathscr{A}^\bullet(S,\mathbf{V})$ is a fine resolution of $\Omega^\bullet_S(\mathbf{V})$, possessing a corresponding filtration. We summarize the main consequences:

(5.9) PROPOSITION. *Assume that* S *is compact. Then*

i) *The spectral sequence*

$$E_1^{p,q} = \mathbf{H}^{p+q}(S, Gr^p_F\,\Omega^\bullet_S(\mathbf{V})) \Rightarrow \mathbf{H}^{p+q}(S, \Omega^\bullet_S(\mathbf{V})) \simeq H^{p+q}(S, \mathbf{V})$$

degenerates at E_1.

ii) *The filtration induced by* $\{F^r\,\Omega^\bullet_S(\mathbf{V})\}$ *on* $H^n(S, \mathbf{V})$ *coincides with the Hodge filtration associated to the decomposition of (5.6) under Definition (4.4).*

iii) *There is a natural identification*

$$H^{P,Q} \simeq \mathbf{H}^n(S, Gr^P_F\,\Omega^\bullet_S(\mathbf{V})$$

for $P+Q=m+n$.

Proof. The above statements are all immediate consequences of (5.6).

We now specialize to the case of a locally homogeneous variation of Hodge structure associated to (ρ, V), as described in the preceding section. First, we recall the differential operators from (3.9) and (3.10), and observe:

(5.10) PROPOSITION. *For a locally homogeneous variation of Hodge structure,*

$$\partial' = D' \quad \text{and} \quad \nabla' = d'_\rho .$$

(5.11) COROLLARY. *Under the same hypothesis,*

$$\mathfrak{D}' = D' + d''_\rho \quad \text{and} \quad \mathfrak{D}'' = D'' + d'_\rho .$$

This, when coupled with (5.3), explains (3.19).

We seem to have two different Hodge decompositions on $\bar{H}^n_{(2)}(S, \mathbf{V})$, one (5.6) coming from the variation of Hodge structure, and one given by (3.18). The two, in fact, are mutually compatible, for (3.14) implies that the Laplacian $\square_d$ respects the complete decomposition (5.1). Thus, we obtain in the locally homogeneous case

(5.12) $$\mathscr{h}^n_{(2)}(S, \mathbf{V}) = \bigoplus_{\substack{p+q=n \\ r+s=m}} \mathscr{h}^{(p,q);(r,s)}_{(2)}(S, \mathbf{V}),$$

and thus

(5.13) $$\bar{H}^n_{(2)}(S, \mathbf{V}) = \bigoplus_{\substack{p+q=n \\ r+s=m}} \bar{H}^{(p,q);(r,s)}_{(2)}(S, \mathbf{V}),$$

with

(5.14) $$\bar{H}^{p,q}_{(2)}(S, \mathbf{V}) = \bigoplus_{r+s=m} \bar{H}^{(p,q);(r,s)}_{(2)}(S, \mathbf{V}),$$

(5.15) $$\bar{H}^{P,Q}_{(2)} = \bigoplus_{\substack{p+r=P \\ q+s=Q}} \bar{H}^{(p,q);(r,s)}_{(2)}(S, \mathbf{V}).$$

As a consequence of (5.12), we derive

(5.16) PROPOSITION. *Assume S is compact. Then for all integers k, the spectral sequence*

$$_IE_1^{p,q} = H^q(S, \Omega^p_S \otimes \mathscr{G}_{\mathfrak{z}}^{\,k-p}) \Rightarrow \mathbf{H}^{p+q}(S, Gr^k_F \Omega^\bullet_S(\mathbf{V}))$$

degenerates at E_2.

Proof. We have the isomorphism

$$\mathbf{H}^n(S, Gr_F^k \Omega_S^\bullet(\mathbf{V})) \simeq H^{k,\,n+m-k} = \bigoplus_{\substack{p+r=k \\ q+s=n+m-k}} H^{(p,q);\,(r,s)}(S, \mathbf{V}),$$

and by the Hodge-Dolbeault isomorphism and [11, (1.10)]

$$ {}_I E_1^{p,q} = H^q(S, \Omega_S^p \otimes \mathcal{G}_{\mathfrak{k}}^{k-p}) \simeq \{\eta \in \mathcal{A}^{p,q}(\mathcal{H}^{k-p,\,m-k+p}) : \Box_{\bar\partial'}\eta = 0\}.$$

The E_2 term of the spectral sequence is equal to the cohomology of the E_1 term under its differential d_1. We again use [11, (1.10)] to assert that in terms of the $\bar\partial'$-harmonic forms, d_1 is given by ∇'. In other words, the E_2 terms are naturally isomorphic to the cohomology groups of the complex of D''-harmonic forms under the differential d_ρ'. Representing classes by d_ρ'-harmonic forms, we have

$$\bigoplus {}_I E_2^{p,q} \simeq \{\eta : \Box_{D''}\eta = 0, \quad \text{and also} \quad \Box_{d_\rho'}\eta = 0\}.$$

By (3.20), the right-hand side gives $H^{k,\,n+m-k}$, so the desired conclusion follows.

(5.17) COROLLARY 1. *There is a natural injection*

$$\mathcal{h}^{(p,q);\,(r,s)} \to H^q(S, \Omega_S^p \otimes \mathcal{G}_{\mathfrak{k}}^{r})$$

(5.18) COROLLARY 2 [7, p. 413]. $H^{0,n}(S, \mathbf{V}) = H^{(0,n);\,(0,m)}(S, \mathbf{V})$. *(Hence also* $H^{n,0}(S, \mathbf{V}) = H^{(n,0);\,(m,0)}(S, \mathbf{V})$.*)*

Proof. Let η be a harmonic $(0, n)$-form with values in $\mathbf{V}$. Then by (3.20), we must have

$$d_\rho'\eta = 0.$$

Since η is an anti-holomorphic form, we than have

$$\rho(X)\eta = 0 \quad \text{for all} \quad X \in \mathfrak{p}^+.$$

This forces the form $\tilde\eta$ (3.3) to take its values in $V\langle m\rangle$, proving (5.18) by the last assertion in §4.

We also have:

(5.19) PROPOSITION. *The spectral sequence*

$$ {}_{II}E_2^{p,q} = H^p(S, \mathcal{H}^q(Gr_F^k\Omega_S^\bullet(\mathbf{V})) \Rightarrow \mathbf{H}^{p+q}(S, Gr_F^k\Omega_S^\bullet(\mathbf{V}))$$

degenerates at E_2 *if* S *is compact, and*

$$_{II}E_2^{p,q} \simeq H^{(q,p);(k-q,m+k-q)}(S, \mathbf{V}).$$

Proof. Let

$$\begin{aligned}
\mathscr{H}_k^q &= \mathscr{H}^q(Gr_F^k \Omega_S^\bullet(\mathbf{V})),\\
\mathscr{C}_k^q &= \Omega_S^q \otimes \mathscr{G}_{\mathfrak{z}}^{k-q},\\
\mathscr{Z}_k^q &= \ker\{\Omega_S^q \otimes \mathscr{G}_{\mathfrak{z}}^{k-q} \to \Omega_S^{q+1} \otimes \mathscr{G}_{\mathfrak{z}}^{k-q-1}\}\\
\mathscr{B}_k^q &= \operatorname{im}\{\Omega_S^{q-1} \otimes \mathscr{G}_{\mathfrak{z}}^{k-q+1} \to \Omega_S^q \otimes \mathscr{G}_{\mathfrak{z}}^{k-q}\}.
\end{aligned}$$

Then all of the above are automorphic vector bundles associated to representations of K on vector spaces H_k^q, C_k^q, Z_k^q, B_k^q respectively. As a consequence of Schur's Lemma and the semi-simplicity of K-representations,

$$C_k^q \simeq H_k^q \oplus B_k^q \oplus B_k^{q+1}$$

as a representation of K. Therefore, by (2.7),

$$\mathscr{C}_k^q \simeq \mathscr{H}_k^q \oplus \mathscr{B}_k^q \oplus \mathscr{B}_k^{q+1}.$$

This implies that there is an embedding

$$\oplus_q \mathscr{H}_k^q \to Gr_F^k \Omega_S^\bullet(\mathbf{V}),$$

which is a quasi-isomorphism. From this, it is clear that the spectral sequence $_{II}E_2^{p,q}$ must degenerate at E_2, and moreover that $H^p(S, \mathscr{H}_k^q)$ gives the (q,p); $(k-q, m+q-k)$ component of $H^{p+q}(S, \mathbf{V})$.

(5.20) *Remark.* We also obtain from the above that

$$_{I}E_2^{p,q} \simeq {}_{II}E_2^{q,p},$$

so the argument of (5.19) gives an alternate proof of (5.16).

By combining (5.19) with (5.9), we obtain

(5.21) COROLLARY. *If* S *is compact,*

$$\dim H^n(S, \mathbf{V}) = \sum_k \sum_{p+q=n} \dim H^p(S, \mathscr{H}_k^q).$$

We can generalize (5.19) to the non-compact case, if we forego the hypercohomology. Because a morphism of representations of K induces a *bounded* mapping between the associated locally homogeneous vector bundles, we can see, by reasoning similar to that used in (5.19), that

$$(5.22)\qquad H^{q,\,p;\,k-q,\,m+q-k}_{(2)}(S,\mathbf{V}) \simeq {}_{\bar\partial}H^{p}_{(2)}(S,\mathscr{H}^{q}_{k})$$

$$=: \frac{\{\phi \in \mathscr{A}^{0,\,p}_{(2)}(\mathscr{H}^{q}_{k}) : \bar\partial\phi = 0\}}{\{\phi \text{ as above: } \phi = \bar\partial\eta \text{ for some } \eta \in \mathscr{A}^{0,\,p-1}_{(2)}(\mathscr{H}^{q}_{k})\}}$$

If we make use of the full extent of (3.29), we can actually deduce the following generalization of (5.9):

(5.23) THEOREM. *Let $Y^{\bullet}$ be a Q-invariant $d'_{\mathfrak{p}}$-subcomplex of $\Lambda^{\bullet}\mathfrak{p}^{-} \otimes V$ (Q as in (1.10)), such that $\mathfrak{p}^{-} \wedge Y^{p} \subset Y^{p+1}$, so that a sub-complex $\mathscr{Y}^{\bullet}$ of holomorphic sub-bundles of $\Omega^{\bullet}_{S}(\mathbf{V})$ is determined by $Y^{\bullet}$. Assume also that*

$$Y^{\bullet} \cap d'_{\mathfrak{p}}(\Lambda^{\bullet}\mathfrak{p}^{-} \otimes V) = d'_{\mathfrak{p}}\, Y^{\bullet}.$$

Then if S is compact, there are short exact sequences

$$0 \to \mathbf{H}^{n}(S,\mathscr{Y}^{\bullet}) \xrightarrow{\iota} H^{n}(S,\mathbf{V}) \to \mathbf{H}^{n}(S,\Omega^{\bullet}_{S}(\mathbf{V})/\mathscr{Y}^{\bullet}) \to 0$$

for all n, with $\iota\left(\mathbf{H}^{n}(S,\mathscr{Y}^{\bullet})\right)$ given by the subspace of harmonic n-forms with values in $\mathscr{Y}^{\bullet}$.

Proof. Let $F^{p}\mathscr{Y}^{\bullet}$ be the filtration on $\mathscr{Y}$ induced by (5.7). Consider the spectral sequences

$${}_{A}E^{p,\,q}_{1} = \mathbf{H}^{p+q}(S, Gr^{p}_{F}\mathscr{Y}^{\bullet}) \Rightarrow \mathbf{H}^{p+q}(S,\mathscr{Y}^{\bullet}),$$

$${}_{B}E^{r,\,s}_{1} = \mathbf{H}^{s}(S, Gr^{p}_{F}\mathscr{Y}^{r}) \Rightarrow \mathbf{H}^{r+s}(S, Gr^{p}_{F}\mathscr{Y}^{\bullet}).$$

As in (5.16), the second one degenerates at E_2, with

$${}_{B}E^{r,\,s}_{2} \simeq H^{s}(S, \mathscr{H}^{r} Gr^{p}_{F}\mathscr{Y}^{\bullet}.$$

By assumption, $\mathscr{H}^{p}\, Gr^{p}_{F}\mathscr{Y}^{\bullet}$ may be identified with an equivariant *sub-bundle* of $\mathscr{H}^{r}_{p}$, whence ${}_{B}E^{r,\,s}_{2}$ becomes identified with a subspace of $\mathscr{h}^{(r,\,s);\,(p-r,\,m-p+r)}(S,\mathbf{V})$. Thus, the mapping

$${}_{A}E^{p,\,q}_{1} \to \mathbf{H}^{p+q}(S, Gr^{p}_{F}\Omega^{\bullet}_{S}\;(\mathbf{V}))$$

is an injection. Since the spectral sequence (5.9, i) degenerates at E_1, it follows that ${}_{A}E^{p,\,q}$ does likewise, and the assertions of (5.23) follow.

(5.24) *Remark.* Take $\mathscr{Y}^{\bullet} = (F^{p}\Omega^{\bullet}_{S}) \otimes \mathbf{V}$ (any p) in (5.23). Then one recovers (3.18) and its algebraic consequences: the spectral sequence

$$E^{p,\,q}_{1} = H^{q}(S,\Omega^{p}_{S}(\mathbf{V})) \Rightarrow \mathbf{H}^{p+q}(S,\Omega^{\bullet}_{S}(\mathbf{V})) \simeq H^{p+q}(S,\mathbf{V})$$

degenerates at E_1.

We next analyze the terms of the Hodge decomposition, as given in (5.9, iii). In particular, we will concern ourselves with the vanishing of some of these terms.

Given the irreducible representation (ρ, V) of G, we let τ_p denote the representation of K on the subspace $H_0^{p,q}$ (4.9) of V obtained by restricting ρ. The following is an immediate consequence of our constructions:

(5.25) LEMMA. *There is a holomorphic isomorphism*

$$\mathcal{G}\ell^{p} \simeq E(\Gamma, \tau_p).$$

(5.26) COROLLARY. *As holomorphic vector bundles on S, the terms of the complex $Gr_F^k\,\Omega_S^\bullet(\mathbf{V})$ are*

$$\Omega_S^p \otimes \mathcal{G}\ell^{k-p} \simeq E(\Gamma, \Lambda^p \mathrm{Ad}^- \otimes \tau_{k-p}).$$

(5.27) COROLLARY. *Assume S is compact. Then for all n, $H^{n,0}(S, \mathbf{V})$ is given by the space of automorphic forms*

$$\{f \in \Gamma(M, \Omega_M^0) \otimes \Lambda^n \mathfrak{p}^- \otimes H_0^{m,0} : f(\gamma x) = (\Lambda^n \mathrm{Ad}^- \otimes \tau_m)\big(\mathcal{J}(\gamma, x)\big) \cdot f(x)\}$$

Proof. Combine (5.18) and (5.26) with (2.13).

Establishing the vanishing of some of the $H^{P,Q}$ is easiest if we can prove that the complex $Gr_F^P\,\Omega_S^\bullet(\mathbf{V})$ is acyclic, or is at least close to being so (cf. [11, §12]). Since the differentials in this complex are $\mathcal{O}_S$-linear, we are reduced to a problem of linear algebra. We make the following simple observation:

(5.28) LEMMA. *Under the identification (5.26), the differentials in $Gr_F^p\,\Omega_S^\bullet(\mathbf{V})$ are given by d_ρ' (3.10).*

As was pointed out to me by David DeGeorge, the operator d_ρ', when applied to *all* of $\Lambda^\bullet \mathfrak{p}^- \otimes V$, gives rise to the Lie algebra cohomology $H^\bullet(\mathfrak{p}^+, V)$ for the Abelian Lie algebra $\mathfrak{p}^+$. We have the C^∞ isomorphism

$$E(\Gamma, \Lambda^\bullet \mathrm{Ad}^- \otimes \rho) \simeq \bigoplus_k Gr_F^k\,\Omega_S^\bullet(\mathbf{V}).$$

Moreover, we can recover each summand $Gr_F^k\Omega_S^\bullet(\mathbf{V})$, since the central subgroup Δ (defined after (1.8)) of K acts on $\Lambda^p \mathfrak{p}^- \otimes H^{k-p,\,m-k+p}$ by the character

$$\chi_{-p\mu + [\lambda - (k-p)\mu]} = \chi_{\lambda - k\mu}$$

independent of p, and faithfully determined by k.

Since $d'_{\mathfrak{p}}$ commutes with the action of K (see, e.g., [10, (2.5.1.1)]), $H^q(\mathfrak{p}^+, V)$ is a representation space for K, and equals the direct sum of certain irreducibles contained in the q-cochains (cf. the proof of (5.19)). Let σ_l^q denote the representation of K on $H^q(\mathfrak{p}^+, V) < l >$. Since the construction

$$\tau \mapsto E(\Gamma, \tau)$$

is an exact functor, we combine the above statements to obtain

(5.29) THEOREM. *The cohomology sheaves of* $Gr_F^k \Omega_S^\bullet(\mathrm{V})$ *are given by*

$$\mathscr{H}_k^q \simeq E(\Gamma, \sigma_{\lambda-k\mu}^q).$$

(5.30) *Remark.* This explains the occurrence of Lie algebra cohomology in [8, §10]. Also, we point out that (5.18) can be deduced from (5.29).

In order to compute the cohomology sheaves in (5.29), we will make use of a result due to Kostant [6]. Let $\mathfrak{h}$ be a Cartan sub-algebra of $\mathfrak{g}_{\mathbf{C}}$, which we may take to be contained in $\mathfrak{k}_{\mathbf{C}}$. Let Ψ denote the set of roots for $\mathfrak{g}_{\mathbf{C}}$ relative to $\mathfrak{h}$. Let Ψ_1 denote the set of compact roots, i.e., the roots α for which the associated eigenspace $\mathfrak{g}_\alpha \subset \mathfrak{g}_{\mathbf{C}}$ is contained in $\mathfrak{k}_{\mathbf{C}}$; let Ψ_2 denote the set of complementary (non-compact) roots. It is possible to choose the positive Weyl chamber in $\mathfrak{h}^*$ so that for the set of positive complementary roots Ψ_2^+,

$$\mathfrak{p}^+ = \bigoplus_{\alpha \in \Psi_2^+} \mathfrak{g}_\alpha .$$

Then

$$\mathfrak{q} = \mathfrak{k}_{\mathbf{C}} \oplus \mathfrak{p}^+$$

is a parabolic subalgebra of $\mathfrak{g}_{\mathbf{C}}$, with $\mathfrak{p}^+$ its nilpotent radical, and $\mathfrak{k}_{\mathbf{C}}$ a Levi subalgebra of $\mathfrak{q}$. We also define Ψ^+, Ψ^-, and Ψ_1^+ in the obvious way. Then for our set-up, the theorem of [6] states

(5.31) THEOREM (Kostant). *For a dominant integral weight* $\Lambda \in \mathfrak{h}^*$, *let* V_Λ *be the irreducible representation of* G *with highest weight* Λ. *Then as a representation of* K,

$$H^q(\mathfrak{p}^+, V_\Lambda) \simeq \bigoplus_{w \in W_u(q)} E_{w(\Lambda+\delta)-\delta}$$

where $\delta = \frac{1}{2}\sum_{\alpha \in \Psi^+} \alpha$, E_β *is the representation of* K *with highest weight* β, *and* $W_u(q)$ *denotes the subset of the Weyl group for* $\mathfrak{h}$ *consisting of those* *elements* w *which move exactly* q *elements of* Ψ^- *into* Ψ_2^+, *but no elements of* Ψ^- *into* Ψ_1^+. [1])

(5.32) COROLLARY. $\mathscr{H}_k^q$ *is a holomorphic vector bundle with fiber isomorphic to*

$$\bigoplus_{w \in W_u(q)} E_{w(\Lambda+\delta)-\delta} <\lambda - k\mu> .$$

Since E_β is an irreducible representation of K, the subgroup Δ of K acts with a single character $\chi_{n(\beta)}$, which may be determined from the highest weight itself. In fact, it is clear that $\beta \mapsto n(\beta)$ extends to a linear functional on $\mathfrak{h}^*$, given by evaluation on a uniquely determined element of $\mathfrak{z}$. (This point will also be used in the discussion of vanishing theorems at the end of this section.) Therefore, we may rewrite the terms in the formula of (5.32) as

$$E_{w(\Lambda+\delta)-\delta} <\lambda - k\mu> = \begin{cases} E_{w(\Lambda+\delta)-\delta} & \text{if } \lambda - k\mu = n[w(\Lambda+\delta)-\delta] \\ 0 & \text{otherwise}. \end{cases}$$

(5.33) *Example.* In the case $G = SL(2, \mathbf{R})$, $\Psi_1 = \emptyset$, $\mathfrak{h}$ is one-dimensional, and the highest weights can be identified with the non-negative integers. Let $V_m = \mathrm{Symm}^m(\mathbf{C}^2)$, and let ρ_m be the corresponding representation of G. Then $\rho_2 \simeq \mathrm{Ad}$, so under the identification with integers, $\delta = 1$. Moreover, the Weyl group decomposes as the identity element I in $W_u(0)$, and $-I$ in $W_u(1)$. Thus, we obtain

$$\mathscr{H}^0 (Gr_F^k \Omega_S^\bullet(\mathbf{V}_m)) = E_m < m - 2k >$$
$$\mathscr{H}^1 (Gr_F^k \Omega_S^\bullet(\mathbf{V}_m)) \simeq E_{-m-2} < m - 2k > ,$$

as $\mu = 2$. But for $SL(2, \mathbf{R})$, $Z = K$, and therefore

$$E_m < n > = \begin{cases} E_m & \text{if } n = m \\ 0 & \text{if } n \neq m. \end{cases}$$

Inserting this above, we see that $Gr_F^k \Omega_S^\bullet(\mathbf{V}_m)$ is acyclic except for $k = 0$ (where $\mathscr{H}^0 \neq 0$) and $k = m + 1$ (where $\mathscr{H}^1 \neq 0$); this yields the Shimura isomorphism, cf. [11, (12.14)].

The above gives rise to an interesting approach to cohomology vanishing theorems for real representations, like those of [8, Thms. (8.2), (12.1)]—there, however, no realness hypothesis is imposed on the representation. The idea is

[1]) $\cup_q W(q)$ is a set of representatives for the Weyl group of $\mathfrak{g}_{\mathbf{C}}$ modulo that of $\mathfrak{k}_{\mathbf{C}}$, consisting of those elements which keep the positive chamber for $\mathfrak{g}_{\mathbf{C}}$ inside the larger positive chamber for $\mathfrak{k}_{\mathbf{C}}$.

relatively simple: if cohomology occurs in multi-degree $(p, q; k, m-k)$, then (by conjugation) it must also occur in multi-degree $(q, p; m-k, k)$.

(5.34) PROPOSITION. *Let* Λ *be the highest weight of the real representation* (ρ, V) *of the group* G. *Then a necessary condition that* $H^{p,q}(S, \mathbf{V}) \neq 0$ *is that there exist* $w_1 \in W_u(p)$ *and* $w_2 \in W_u(q)$ *such that* $n[w_1\Lambda + w_2\Lambda] = 0$.

Proof. By (5.32),

$$H^{(p,q);(k-p,m+p-k)}(S, \mathbf{V}) \simeq H^q(S, \mathscr{H}^p_k) = 0$$

unless there exists $w_1 \in W_u(p)$ with

$$n[w_1(\Lambda+\delta)-\delta] = \lambda - k\mu. \tag{5.35}$$

(Note that $\lambda = n(\Lambda)$ and $\mu = n(\Xi)$, where Ξ denotes the highest weight of Ad.)

Similarly, for the conjugate term

$$H^{(q,p);(m+p-k,k-p)}(S, \mathbf{V}) \simeq H^p(S, \mathscr{H}^q_{m+p+q-k})$$

to be non-zero, we must also have for some $w_2 \in W_u(q)$

$$n[w_2(\Lambda+\delta)-\delta] = \lambda - (m+p+q-k)\mu. \tag{5.36}$$

Adding (5.35) and (5.36), we see that

$$\dim H^q(S, \mathscr{H}^p_k) = \dim H^p(S, \mathscr{H}^q_{m+p+q-k}) = 0$$

unless there exist $w_1 \in W_u(p)$ and $w_2 \in W_u(q)$ such that

$$n[w_1(\Lambda+\delta)-\delta] + n[w_2(\Lambda+\delta)-\delta] = 2\lambda - (m+p+q)\mu. \tag{5.37}$$

We use the identities

$$n[\delta - w\delta] = q\mu \quad \text{if} \quad w \in W_u(q),$$
$$2\lambda = m\mu$$

to rewrite (5.37) as

$$n[w_1\Lambda] + n[w_2\Lambda] = 0, \tag{5.38}$$

or

$$n[w_1\Lambda + w_2\Lambda] = 0.$$

Let $\zeta = \sum_{\alpha \in \Psi_2^+} \alpha$. It is easily checked that under the isomorphism $\mathfrak{h}^* \simeq \mathfrak{h}$ via the Killing form, ζ represents a non-zero element of $\mathfrak{z}$. Thus, the condition (5.38) can be rewritten as

$$<w_1\Lambda, \zeta> + <w_2\Lambda, \zeta> = 0,$$

or

$$<\Lambda, w_1^{-1}\zeta + w_2^{-1}\zeta> = 0. \tag{5.39}$$

We are discussing real representations, for which $w_0\Lambda = -\Lambda$ (by self-contragredience) with w_0 denoting the (unique) element of the Weyl group which maps Ψ^+ to Ψ^-. It suffices then to examine the sums

$$w_1^{-1}\zeta - w_0 w_2^{-1}\zeta \qquad w_1 \in W_u(p), w_2 \in W_u(q).$$

For instance, the assertion that $H^{p,q}(\Gamma; \rho, V) = 0$ if $p + q \neq \dim S$ and Λ lies in the interior of the dominant cone [8, Thm. (12.1)] would follow from the corresponding statement: $w_1^{-1}\zeta - w_0 w_2^{-1}\zeta$ is a non-zero element of the positive dual cone for all $w_1 \in W_u(p)$ and $w_2 \in W_u(q)$ whenever $p + q < \dim S$. This has been verified by the author in some examples, though no satisfying argument involving root structure has been found.

Added in proof: Borel has pointed out that although there was a gap in the proof of Theorem 4 of [13], it has been filled in by W. Casselman in the case where the ranks of G and K are equal. This includes all Hermitian cases.