II. Preliminaries

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Let S be a connected, locally noetherian scheme, and s a geometric point of S (i.e., s is a point of S with values in an algebraically closed field). The fundamental group $\pi_1(S, s)$ in the sense of SGA I is a profinite group which classifies the finite etale coverings of S. Given two geometric points s_1 and s_2 each choice of "chemin" $c(s_1, s_2)$ from s_1 to s_2 determines an isomorphism

$$c(s_1, s_2): \pi_1(S, s_1) \xrightarrow{\sim} \pi_1(S, s_2)$$

and formation of this isomorphism is compatible with composition of chemins. If we fix s_1 and s_2 but vary the chemin, this isomorphism will (only) change by an inner automorphism of, say, $\pi_1(S, s_2)$.

Therefore the *abelianization* of $\pi_1(S, s)$ (in the category of profinite groups) is canonically independant of the auxiliary choice of base point; we will denote it $\pi_1(S)^{ab}$. This profinite abelian group classifies (*fppf*) torsors over S with (variable) finite abelian structure group, i.e. for any finite abelian group G we have a canonical isomorphism

$$\text{(1.1)} \qquad \mathrm{Hom}_{gp}(\pi_1(S)^{ab}, G) \xrightarrow{\sim} H^1_{et}(S, G).$$

The total space of the G-torsor T/S is connected if and only if its classifying map $\pi_1(S)^{ab} \to G$ is surjective.

Given a morphism $f: X \to S$ between connected locally noetherian schemes, a geometric point x of X and its image $s = f(x)$ in S, there is an induced homomorphism

$$\pi_1(X, x) \to \pi_1(S, s)$$

of fundamental groups. The induced homomorphism

$$\pi_1(X)^{ab} \to \pi_1(S)^{ab}$$

is independent of the choice of geometric point x; indeed for any finite abelian group G the transposed map

$$\mathrm{Hom}\left(\pi_1(S)^{ab}, G\right) \to \mathrm{Hom}\left(\pi_1(X)^{ab}, G\right)$$

is naturally identified with the map "inverse image of G-torsors"

$$f^*: H^1_{et}(S; G) \to H^1_{et}(X; G).$$

We will denote by Ker (X/S) the kernel of the map of π_1^{ab}'s. Thus we have a tautological exact sequence

(1.2) $$0 \to \mathrm{Ker}\,(X/S) \to \pi_1\,(X)^{ab} \to \pi_1\,(S)^{ab}\,.$$

When X/S has a section

$$\begin{tikzcd} X \arrow[d, "f"'] \\ S \arrow[u, bend right, "\varepsilon"'] \end{tikzcd}$$

there is a simple interpretation of Ker (X/S); it classifies those torsors on X with finite abelian structure group whose inverse image via ε is trivial on S, i.e. whose restriction to the section, viewed as a subscheme of X, is completely decomposed. There is a natural product decomposition

(1.3) $$\pi_1\,(X)^{ab} \simeq \pi_1\,(S)^{ab} \times \mathrm{Ker}\,(X/S)$$

corresponding to the expression of a G-torsor on X as the "sum" of a G-torsor on X whose restriction to ε is completely decomposed and the inverse image by f of a G-torsor on S. In particular, given, a G-torsor T/X whose restriction to ε is completely decomposed, T is connected if and only if its classifying map Ker $(X/S) \to G$ is surjective. In the absence of a section, there seems to be no simple physical interpretation of Ker (X/S).

There are two elementary functorialities. it is convenient to formulate explicitly. Consider a commutative diagram of morphisms of connected, locally noetherian schemes

$$\begin{tikzcd} & Y \arrow[dl] \arrow[dd] \\ X \arrow[dd] & \\ & T \arrow[dl] \\ S & \end{tikzcd}$$

Proceeding down to the left, we have an exact sequence

(1.4) $$0 \to \mathrm{Ker}\,(Y/X) \to \mathrm{Ker}\,(Y/S) \to \mathrm{Ker}\,(X/S)\,.$$

Proceeding across, we have an induced map

(1.5) $$\mathrm{Ker}\,(Y/T) \to \mathrm{Ker}\,(X/S)$$

which sits in a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \operatorname{Ker}(Y/T) & \longrightarrow & \pi_1(Y)^{ab} & \longrightarrow & \pi_1(T)^{ab} \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \operatorname{Ker}(X/S) & \longrightarrow & \pi_1(X)^{ab} & \longrightarrow & \pi_1(S)^{ab}. \end{array}$$

Let X be a geometrically connected noetherian scheme over a field K, (i.e. $X \otimes \overline{K}$ is connected, where $\overline{K}$ denotes an algebraic closure of K). Let $\bar{x}$ be a geometric point of $X \otimes \overline{K}$, x its image in X, and s its image in $\operatorname{Spec}(K) = S$. The fundamental exact sequence (SGA I, IX, 6.1)

$$0 \to \pi_1(X \otimes \overline{K}, \bar{x}) \to \pi_1(X, x) \to \pi_1(S, s) \to 0 \tag{1.6}$$

yields, upon abelianization, an exact sequence

$$\pi_1(X \otimes \overline{K})^{ab} \to \pi_1(X)^{ab} \to \pi(S)^{ab} \to 0 \tag{1.7}$$

The exact sequence (1.6) allows us to define an action "modulo inner automorphism" of $\pi_1(S, s)$ on $\pi_1(X \otimes \overline{K}, \bar{x})$ (given an element $\sigma \in \pi_1(S, s)$, choose $\tilde{\sigma} \in \pi_1(X, x)$ lying over it and conjugate $\pi_1(X \otimes \overline{K}, \bar{x})$ by this $\tilde{\sigma}$). The induced action of $\pi_1(S, s)$ on $\pi_1(X \otimes \overline{K})^{ab}$ is therefore well-defined. (This same action is well-defined, and trivial, on $\pi_1(X)^{ab}$.)

Therefore the map $\pi_1(X \otimes \overline{K})^{ab} \to \pi_1(X)^{ab}$ factors through the coinvariants of the action of $\pi_1(S, s)$ on $\pi_1(X \otimes \overline{K})^{ab}$: we have an exact sequence

$$(\pi_1(X \otimes \overline{K})^{ab})_{\pi_1(S, s)} \to \pi_1(X)^{ab} \to \pi_1(S)^{ab} \to 0. \tag{1.8}$$

If we identify $\pi_1(S, s)$ for $S = \operatorname{Spec}(K)$ with the galois group $\operatorname{Gal}(\overline{K}/K)$, (which we may do canonically (only) up to an inner automorphism), then this last exact sequence may be rewritten

$$(\pi_1(X \otimes \overline{K})^{ab})_{\operatorname{Gal}(\overline{K}/K)} \to \pi(X)^{ab} \to \operatorname{Gal}(\overline{K}/K)^{ab} \to 0. \tag{1.9}$$

Consider the special case in which X has a K-rational point x_0; if we choose for $\bar{x}$ the geometric point "x_0 viewed as having values in the overfield $\overline{K}$ of K" then the morphism $x_0\colon \operatorname{Spec}(K) \to X$ which "is" x_0 gives a splitting of the exact sequence (1.6)

$$0 \to \pi_1(X \otimes \overline{K}, \bar{x}) \to \pi_1(X, x) \underset{\overset{\curvearrowleft}{x_0}}{\to} \pi_1(S, s) \to 0 \tag{1.10}$$

so that we have a semi-direct product decomposition

$$\pi_1(X, x) \simeq \pi_1(X \otimes \overline{K}, \bar{x}) \ltimes \operatorname{Gal}(\overline{K}/K). \tag{1.11}$$

"Physically", the action of $\mathrm{Gal}(\overline{K}/K)$ on $\pi_1(X \otimes \overline{K}, \bar{x})$ is simply induced by the action of $\mathrm{Gal}(\overline{K}/K)$ on the coefficients of the defining equations of finite etale coverings of $X \otimes \overline{K}$; this action is well defined on $\pi_1(X \otimes \overline{K}, \bar{x})$ precisely because $\bar{x}$ is a $\overline{K}$-valued point which is fixed by $\mathrm{Gal}(\overline{K}/K)$; if $\bar{x}$ were not fixed, an element $\sigma \in \mathrm{Gal}$ would "only" define an isomorphism

$$\pi_1(X \otimes \overline{K}, \bar{x}) \xrightarrow{\sim} \pi_1(X \otimes \overline{K}, \sigma(\bar{x})).$$

The semi-direct product decomposition (1.11) yields, upon abelianization, a product decomposition

(1.12) $$\pi_1(X)^{ab} \xrightarrow{\sim} ((\pi_1(X \otimes \overline{K})^{ab})_{\mathrm{Gal}(\overline{K}/K)} \times \mathrm{Gal}(\overline{K}/K)^{ab};$$

in other words, the existence of a K-rational point on X assures that the right exact sequence (1.9) is actually a split short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & ((\pi_1(X \otimes \overline{K})^{ab})_{\mathrm{Gal}(\overline{K}/K)} \arrow[r] & \pi_1(X)^{ab} \arrow[r] & \mathrm{Gal}(\overline{K}/K)^{ab} \arrow[r] \arrow[l, bend right] & 0. \end{tikzcd}$$

For ease of later reference, we explicitly formulate the following lemma.

Lemma 1. *Let X be a geometrically connected noetherian scheme over a field K. Then* $\mathrm{Ker}(X/K)$ *is the image of* $\pi_1(X \otimes \overline{K})^{ab}$ *in* $\pi_1(X)^{ab}$. *The natural surjective homomorphism*

$$\pi_1(X \otimes \overline{K})^{ab} \to \mathrm{Ker}(X/K)$$

factors through a surjection

(1.14) $$(\pi_1(X \otimes \overline{K})^{ab})_{\mathrm{Gal}(\overline{K}/K)} \twoheadrightarrow \mathrm{Ker}(X/K)$$

which is an isomorphism if X has a K-rational point. Given any algebraic extension L/K, the natural map

(1.15) $$\mathrm{Ker}(X \underset{K}{\otimes} L/L) \to \mathrm{Ker}(X/K)$$

is surjective.

Proof. The only new assertion is the surjectivity of (1.15), and this follows immediately from the surjectivity of the indicated maps in the commutative diagram

$$\begin{tikzcd}
& (\pi_1 (X \otimes \overline{K})^{ab})_{\mathrm{Gal}\,(\bar{K}/L)} \arrow[r, two heads] \arrow[dd] & \mathrm{Ker}\,(X \underset{K}{\otimes} L/L) \arrow[r, hook] \arrow[dd] & \pi_1 (X \underset{K}{\otimes} L)^{ab} \arrow[dd] \\
\pi_1 (X \otimes \overline{K})^{ab} \arrow[ur] \arrow[dr] & & & \\
& (\pi_1 (X \otimes \overline{K})^{ab})_{\mathrm{Gal}\,(\bar{K}/K)} \arrow[r, two heads] & \mathrm{Ker}\,(X/K) \arrow[r, hook] & \pi_1 (X)^{ab} .
\end{tikzcd}$$

Now consider a normal, connected locally noetherian scheme S with generic point η and function field K. We fix an algebraic closure $\overline{K}$ of K, and denote by $\bar{\eta}$ the corresponding geometric point of S. The fundamental group $\pi_1 (S, \bar{\eta})$ is then a quotient of the Galois group Gal $(\overline{K}/K)$; the functor "fibre over η"

$$\{\text{connected finite etale coverings of } S\} \to \{\text{finite separable extensions } L/K\}$$

is fully faithful, with image those finite separable extensions L/K for which the normalization of S in L is finite etale over S.

LEMMA 2. *Let S be normal, connected and locally noetherian, with generic point η and function field K. Let $f : X \to S$ be a smooth surjective morphism of finite type, whose geometric generic fibre $X_{\bar{\eta}}$ is connected. Then*

(1) *X is normal and connected.*

(2) *For any geometric point $\bar{x}$ in $X_{\bar{\eta}}$ with image x in X and s in S, the sequence*

$$\pi_1 (X_{\bar{\eta}}, \bar{x}) \to \pi_1 (X, x) \to \pi_1 (S, s) \to 0$$

is exact.

(3) Ker (X/S) *is the image of $\pi_1 (X_{\bar{\eta}})^{ab}$ in $\pi_1 (X)^{ab}$.*

(4) *The natural map*

$$\mathrm{Ker}\,(X_\eta/K) \to \mathrm{Ker}\,(X/S)$$

is surjective.

Proof. (1) Because X is smooth over a normal scheme, it is itself normal (SGAI, Exp II, 3.1). To see that X is connected, we argue as follows. The map f, being flat (because smooth) and of finite type over a locally noetherian scheme, is *open* (SGAI, Exp IV, 6.6). Therefore any nonvoid open set $U \subset X$ meets X_η (because $f\,(U)$ is open and non-empty in S, so contains η). But X_η is connected (because $X_{\bar{\eta}}$ is !) and therefore the intersection of any two-non-empty open sets in X meets X_η.

(2) Because X is normal and connected, it has a generic point ξ and a function field F, and its function field F is none other than the function field of X_η (itself normal (because smooth over K) and connected). Therefore the natural map

$$\pi_1(X_{\bar\eta}, \bar\xi) \to \pi_1(X, \bar\xi)$$

must be surjective, because it sits in the commutative diagram

$$\begin{tikzcd}
 & \pi_1(X_\eta, \bar\xi) \arrow[d] \\
\mathrm{Gal}(\bar F/F) \arrow[ur] \arrow[dr] & \\
 & \pi_1(X, \bar\xi)
\end{tikzcd}$$

Comparing our putative exact sequence with its analogue for X_η/K, we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \pi_1(X_{\bar\eta}, \bar x) \arrow[r] \arrow[d, equal] & \pi_1(X_\eta, x) \arrow[r] \arrow[d] & \mathrm{Gal}(\bar K/K) \arrow[r] \arrow[d] & 0 \\
 & \pi_1(X_{\bar\eta}, \bar x) \arrow[r, "\alpha"] & \pi_1(X, x) \arrow[r, "\beta"] & \pi_1(S, s) \arrow[r] & 0
\end{tikzcd}$$

whose top row is exact. Therefore β is surjective, and $\beta \circ \alpha = 0$. To show the exactness, given the surjectivity of β, we must show (cf. SGA I, Exp V, 6.6) that any connected etale covering Y of X which admits a section over $X_{\bar\eta}$ is isomorphic to the inverse image of a connected etale covering of S. Given such Y, its restriction Y_η to X_η is still connected; so the existence of a section over $X_{\bar\eta}$ and the exactness of (1.6) imply that Y_η is the normalization of X_η in a constant-field extension $F \cdot L$, where L is a finite separable extension of K. Therefore the function field of Y is $F \cdot L$, whence Y is the normalization of X in $F \cdot L$. Let S' denote the normalization of S in L. Then S' is finite over S. We will show that S' is finite etale over S, and that Y is the inverse image over X of this covering. By (1) applied to $X \times_S S'/S'$, the scheme $X \times_S S'$ is normal and connected, and finite over X. Therefore $X \times_S S'$ is just the normalization of X in its function field, i.e. in $F \cdot L$. Therefore $Y = X \times_S S'$. It remains only to see that S'/S is finite etale. But this follows by *fpqc* descent from that fact that $Y = X \times_S S'$ is finite etale over X.

(3) This follows immediately from the exact sequence established in (2), by abelianization.

(4) This follows immediately from (3), and the commutativity of the diagram of maps induced by the obvious inclusions

$$\begin{tikzcd} \pi_1(X_\eta)^{ab} \arrow[rr] \arrow[dr] & & \pi_1(X_\eta)^{ab} \arrow[dl] \\ & \pi_1(X)^{ab}. & \end{tikzcd}$$

LEMMA 3. *Let X be a smooth geometrically connected variety of finite type over a field K, and let $U \subset X$ be any non-empty open set. Then the natural map*

$$\mathrm{Ker}(U/K) \to \mathrm{Ker}(X/K)$$

is surjective.

Proof. The variety $X \otimes \overline{K}$ is normal and connected, as is the non-empty open $U \otimes \overline{K}$ in it. Therefore the natural map $\pi_1(U \otimes \overline{K}) \to \pi_1(X \otimes \overline{K})$ is surjective (because both source and target are quotients of the galois group of their common function field). The result now follows from the indicated surjectivities in the commutative diagram

$$\begin{tikzcd} \pi_1(U \otimes \overline{K})^{ab} \arrow[r, two heads] \arrow[d, two heads] & \mathrm{Ker}(U/K) \arrow[d] \\ \pi_1(X \otimes \overline{K})^{ab} \arrow[r, two heads] & \mathrm{Ker}(X/K). \end{tikzcd}$$

II. THE MAIN THEOREM

Recall that a field K is said to be absolutely finitely generated if it is a finitely generated extension of its prime field, i.e. of $\mathbf{Q}$ or of $\mathbf{F}_p$.

THEOREM 1. *Let S be a normal, connected, locally noetherian scheme, whose function field K is an absolutely finitely generated field. Let $f: X \to S$ be a smooth surjective morphism of finite type, whose geometric generic fibre is connected. Then the group* $\mathrm{Ker}(X/S)$ *is finite if K has characteristic zero, and it is the product of a finite group with a pro-p group in case K has characteristic p.*